

# The Resolvent Matrix of the Truncated Hausdorff Matrix Moment Problem via New Dyukarev–Stieltjes Parameters and Extremal Solutions via Continued Fractions

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We obtain a new multiplicative decomposition of the resolvent matrix of the truncated Hausdorff matrix moment (THMM) problem in the case of an odd and even number of moments via new Dyukarev–Stieltjes matrix (DSM) parameters. Explicit interrelations between new DSM parameters and orthogonal matrix polynomials on a finite interval  $[a, b]$ , as well as the Schur complements of the block Hankel matrices constructed through the moments of the THMM problem are given. Additionally, the extremal solutions of the THMM problem are represented via matrix continued fractions in terms of the DSM parameters.

## Keywords:

Resolvent matrix, orthogonal matrix polynomials, Dyukarev-Stieltjes parameters, matrix continued fractions.

## 1. Introduction

Throughout this paper, let  $q$  and  $p$  be positive integers. We will use  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{N}_0$  and  $\mathbb{N}$  to denote the set of all complex numbers, the set of all real numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. The notation  $\mathbb{C}^{q \times q}$  stands for the set of all complex  $q \times q$  matrices. For the null matrix that belongs to  $\mathbb{C}^{p \times q}$  we will write  $0_{p \times q}$ . We denote by  $0_q$  and  $I_q$  the null and the identity matrices in  $\mathbb{C}^{q \times q}$ , respectively. In cases where the sizes of the null and the identity matrix are clear, we will omit the indices.

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The continued fraction

$$\cfrac{1}{m_1 z + \cfrac{1}{l_1 + \cfrac{1}{m_2 z + \cfrac{1}{l_2 + \cfrac{1}{\ddots}}}}} \quad (1.1)$$

where  $l_j$  and  $m_j$  are positive numbers was used by T.J. Stieltjes in [43] to determine whether there is a unique (resp. non-unique) solution of the Stieltjes moment problem. This problem is stated as follows: given a sequence  $(s_k)_{k \geq 0}$  of real numbers, find the set  $\mathcal{M}$  of positive measures  $\sigma$  on  $[0, \infty)$  such that  $s_k = \int_0^\infty x^k d\sigma(x)$  for  $k = 0, 1, \dots$ . Note that

$$l_j := \frac{\Delta_j^2}{\Delta_{j-1}^{(1)} \Delta_{j-2}^{(1)}}, \quad \text{and} \quad m_j := \frac{[\Delta_j^{(1)}]^2}{\Delta_j \Delta_{j-1}}, \quad (1.2)$$

with  $\Delta_j := \det(s_{i+k})_{i,k=0}^j$ ,  $\Delta_j^{(1)} := \det(s_{i+k+1})_{i,k=0}^j$  and  $(\Delta_{-1} = \Delta_{-1}^{(1)} := 1)$ . Here we use Krein's notation [30] for the nonnegative coefficients  $l_j$  and  $m_j$ , also called Stieltjes parameters.

It is well-known that the truncated continued fraction of (1.1) can be written as  $F_p(z, 0)$  where  $F_p$  is a composition of Möbius transformations:  $F_p = f_1 \circ f_2 \circ \dots \circ f_p$  with  $f_{2k-1}(z, \omega) := \frac{1}{-zm_k + \omega}$ ,  $f_{2k}(\omega) := \frac{1}{l_k + \omega}$  for each  $k \in \mathbb{N}$ ; see [31, Page 474]. On the other hand, denoting the matrix of the transformations  $f_{2k-1}$ ,  $f_{2k}$  by  $t_{2k-1} := \begin{pmatrix} 0 & 1 \\ 1 & -zm_k \end{pmatrix}$ ,  $t_{2k} := \begin{pmatrix} 0 & 1 \\ 1 & l_k \end{pmatrix}$ , respectively, one can associate the Möbius transformation with the  $2 \times 2$  matrix

$$\mathcal{T}_{2n} := t_1 t_2 \dots t_{2n}, \quad (1.3)$$

or

$$\mathcal{T}_{2n+1} := t_1 t_2 \dots t_{2n+1} \quad (1.4)$$

for  $\omega = 0$ . See [31, Theorem 12.1a].

In [25] Dyukarev introduced the matrix version of the Stieltjes parameters in order to establish the determinateness of the Stieltjes matrix moment problem. An important feature used in [25] is the similar factorization to (1.3) and (1.4) of the resolvent matrix of the truncated Stieltjes matrix moment problem via Stieltjes parameters. Recall that the mentioned resolvent matrix, also called Nevanlinna matrix [25, Definition 2], is a  $2q \times 2q$  matrix polynomial.

In the present work we introduce new matrix Stieltjes parameters, called Dyukarev-Stieltjes matrix (DSM) parameters of the truncated Hausdorff matrix moment (THMM) problem. With the help of the DSM parameters, we obtain a new multiplicative representation of the resolvent matrix (RM):

$$U^{(m)}(z) = \begin{pmatrix} \alpha^{(m)}(z) & \beta^{(m)}(z) \\ \gamma^{(m)}(z) & \delta^{(m)}(z) \end{pmatrix} \quad (1.5)$$

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of the THMM problem in the case of an odd and even number of moments. The RM  $U^{(m)}$  is a  $2q \times 2q$  matrix polynomial which we factorize as follows:

$$U^{(2n)} = \mathcal{D}_1 \mathbb{I}_{-1}^{(2n)} \mathfrak{m}_0^{(2n)} \dots \mathfrak{m}_{n-1}^{(2n)} \mathbb{I}_{n-1}^{(2n)} \mathcal{B}_2^{(2n)} \mathcal{D}_2 \quad (1.6)$$

$$U^{(2n+1)} = \mathcal{D}_3 \mathbb{I}_{-1}^{(2n+1)} \mathfrak{m}_0^{(2n+1)} \dots \mathbb{I}_{n-1}^{(2n+1)} \mathfrak{m}_n^{(2n+1)} \mathcal{B}_2^{(2n+1)} \mathbf{D}_4 \quad (1.7)$$

where  $\mathcal{D}_k$  are anti-diagonal block matrices,  $\mathbf{D}_4$  is a diagonal matrix,  $\mathcal{B}_2^{(2n)}$ ,  $\mathcal{B}_2^{(2n+1)}$ ,  $\mathbb{I}_j^{(2n+1)}$ ,  $\mathfrak{m}_j^{(2n)}$  are constant anti-triangular block matrices and  $\mathfrak{m}_j^{(2n+1)}$ ,  $\mathbb{I}_j^{(2n)}$  are affine on  $z$  and anti-triangular block matrices; see Theorem 5.2 and Remark 6.1. As a consequence of the representation of the RM as in (1.6) and (1.7), the so-called extremal solutions of the THMM problem  $\alpha^{(2n)} \gamma^{(2n)^{-1}}$ ,  $\beta^{(2n)} \delta^{(2n)^{-1}}$ ,  $\alpha^{(2n+1)} \gamma^{(2n+1)^{-1}}$  and  $\beta^{(2n+1)} \delta^{(2n+1)^{-1}}$  are given by a finite matrix continued fraction; see Theorem 6.1.

Our motivation is to give a complete characterization of solvability of the THMM problem via the DSM parameters. Such characterization will be considered elsewhere. Proposition 7.2 and [9, Proposition 7] are directed to the mentioned characterization. In the indicated propositions, the recovery of positive moment sequences  $(s_j)_{j=0}^m$  (as in Definition 2.1) from the DSM parameters are given; see Definition 5.1.

Let us now summarize the notions appearing in the last two paragraphs.

The THMM problem is stated as follows: given an interval  $[a, b]$  on the real axis and a finite sequence of  $q \times q$  matrices,  $(s_j)_{j=0}^m$ . Describe the set  $\mathcal{M}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^m]$  of all nonnegative Hermitian  $q \times q$  measures  $\sigma$  defined on the  $\sigma$ -algebra of all Borel subsets of the interval  $[a, b]$  such that

$$s_j = \int_{[a, b]} t^j d\sigma(t)$$

holds true for each integer  $j$  with  $0 \leq j \leq m$ .

Let  $(s_j)_{j=0}^{2n}$  (resp.  $(s_j)_{j=0}^{2n+1}$ ) be a sequence of complex  $q \times q$  matrices, then denote

$$H_{1,j} := \tilde{H}_{0,j}, \quad j \geq 0, \quad H_{2,j-1} := -ab\tilde{H}_{0,j-1} + (a+b)\tilde{H}_{1,j-1} - \tilde{H}_{2,j-1}, \quad j \geq 1 \quad (1.8)$$

and

$$K_{1,j} = b\tilde{H}_{0,j} - \tilde{H}_{1,j}, \quad K_{2,j} = -a\tilde{H}_{0,j} + \tilde{H}_{1,j}, \quad j \geq 0, \quad (1.9)$$

where  $\tilde{H}_{\cdot,j}$  are defined with the help of the Hankel matrices,

$$\tilde{H}_{0,j} := \{s_{l+k}\}_{l,k=0}^j, \quad \tilde{H}_{1,j} := \{s_{l+k+1}\}_{l,k=0}^j, \quad \text{and} \quad \tilde{H}_{2,j} := \{s_{l+k+2}\}_{l,k=0}^j.$$

In [10, Theorem 1.3] (resp. [11, Theorem 1.3]), it was demonstrated that there exists a solution to problem  $\mathcal{M}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^{2n}]$  (resp.  $\mathcal{M}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^{2n+1}]$ ) if and only if the block matrices  $H_{1,n}$  and  $H_{2,n-1}$  (resp.  $K_{1,n}$  and  $K_{2,n}$ ) are both nonnegative Hermitian, where the problem of finding the set  $\mathcal{M}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^m]$

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for  $m = 2n$  and  $m = 2n + 1$  is usually reduced to searching for the set of holomorphic functions

$$\mathfrak{S}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^m] := \left\{ s(z) = \int_{[a, b]} \frac{d\sigma(t)}{t - z}, \sigma \in \mathcal{M}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^m] \right\}.$$

If  $H_{1,n}$  and  $H_{2,n-1}$  (resp.  $K_{1,n}$  and  $K_{2,n}$ ) are positive Hermitian, the set  $\mathfrak{S}_{\geq}^q[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^m]$  is parameterized via a linear fractional transformation

$$s(z) = (\alpha^{(m)}(z)\mathbf{p}(z) + \beta^{(m)}(z)\mathbf{q}(z))(\gamma^{(m)}(z)\mathbf{p}(z) + \delta^{(m)}(z)\mathbf{q}(z))^{-1}. \quad (1.10)$$

The column pair  $(\mathbf{p}, \mathbf{q})$  satisfies certain properties in every case; see Definitions [11, Definition 5.2] and [10, Definition 5.2].

The  $2q \times 2q$  polynomial matrices

$$U^{(2j)}(z, a, b) := \begin{pmatrix} \Theta_{2,j}^*(\bar{z}, a)\Theta_{2,j}^{*-1}(a, a) & \frac{1}{b-a}\Theta_{1,j}^*(\bar{z}, b)\Gamma_{1,j}^{*-1}(a, b) \\ (z-a)\Gamma_{2,j}^*(\bar{z}, a)\Theta_{2,j}^{*-1}(a, a) & \frac{b-z}{b-a}\Gamma_{1,j}^*(\bar{z}, b)\Gamma_{1,j}^{*-1}(a, b) \end{pmatrix} \quad (1.11)$$

and

$$U^{(2j+1)}(z, a, b) := \begin{pmatrix} Q_{2,j}^*(\bar{z}, a, b)Q_{2,j}^{*-1}(a, b, a) & -Q_{1,j+1}^*(\bar{z})P_{1,j+1}^{*-1}(a) \\ -(z-a)(b-z)P_{2,j}^*(\bar{z}, a, b)Q_{2,j}^{*-1}(a, b, a) & P_{1,j+1}^*(\bar{z})P_{1,j+1}^{*-1}(a) \end{pmatrix} \quad (1.12)$$

are called the resolvent matrix (RM) of the THMM problem. The  $q \times q$  matrix polynomials  $P_{k,j}$ ,  $Q_{k,j}$ ,  $\Gamma_{k,j}$  and  $\Theta_{k,j}$  for  $k = \{1, 2\}$  are constructed via the given data: the sequence of moments  $(s_j)_{j=0}^{2n}$  (resp.  $(s_j)_{j=0}^{2n+1}$ ). See Definition 2.6 and 2.7.

In [9], the relationships  $U^{(2j+1)} = \begin{pmatrix} \frac{1}{z-a}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \tilde{U}_1^{(2j+1)} A^{(2j+1)} \cdot \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix}$  and  $U^{(2j)} = \tilde{U}_1^{(2j)} A^{(2j)}$  were used, with  $A^{(k)}$  denoting  $2q \times 2q$  matrices depending on  $a$  and  $b$  in order to factorize the matrices  $U^{(m)}$  for  $m$  odd and even. Instead of the mentioned relations, in the present paper we employ the following two relations:

$$U^{(2j)}(z) = \begin{pmatrix} \frac{1}{(z-a)(b-z)}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & s_0 z \\ 0_q & I_q \end{pmatrix} \tilde{U}_2^{(2j-2)}(z) A_2^{(2j)} \begin{pmatrix} (b-a)(z-a)I_q & 0_q \\ 0_q & \frac{b-z}{b-a}I_q \end{pmatrix}. \quad (1.13)$$

and

$$U^{(2j+1)}(z) = \begin{pmatrix} \frac{1}{b-z}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \tilde{U}_2^{(2j+1)}(z) A_2^{(2j+1)} \begin{pmatrix} (b-z)I_q & 0_q \\ 0_q & I_q \end{pmatrix}, \quad (1.14)$$

where  $\tilde{U}_2^{(k)}(z)$ ,  $A_2^{(k)}(z)$  for  $k = 2j$  ( $k = 2j + 1$ ) are introduced in (2.12), (2.14), (2.13) and (2.17). Equalities (1.13) and (1.14) are the consequence of [10, Equality (6.26)] and [11, Equalities (6.26), (6.27)].

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Note that the auxiliary matrices  $\tilde{U}_1^{(2j)}$  and  $\tilde{U}_1^{(2j+1)}$  (resp.  $\tilde{U}_2^{(2j)}$  and  $\tilde{U}_2^{(2j+1)}$ ) are related to  $H_{1,j}$  and  $K_{2,j}$  (resp.  $H_{2,j}$  and  $K_{1,j}$ ), correspondingly.

As in [9, Corollary 1 and Corollary 2] where the auxiliary matrix  $\tilde{U}_1^{(k)}$  was decomposed, in the present work we factorize the auxiliary matrix  $\tilde{U}_2^{(2n+1)}$  in the following form (as in Corollary 4.1):

$$\tilde{U}_2^{(2n+1)} = d^{(1)} d^{(3)} \dots d^{(2n-1)} d^{(2n+1)} \quad (1.15)$$

Instead of  $\tilde{U}_2^{(2n)}$  the auxiliary matrix  $\hat{U}_2^{(2n)}$  (as in (2.15)) is used. The factorization

$$\hat{U}_2^{(2n)} = d^{(0)} d^{(2)} \dots d^{(2n-4)} d^{(2n-2)} \quad (1.16)$$

is employed to prove a new factorization of the RM  $U^{(2n)}$ . The matrices  $d^{(2k+1)}$  and  $d^{(2k)}$  are affine on  $z$ . The importance of the auxiliary matrices  $\tilde{U}_2^{(2j+1)}$  and  $\hat{U}_2^{(2j)}$  resides in the fact that they belong to the Potapov class of matrix functions [39]. The matrix valued functions belonging to this class can be factorized into elementary factors, as seen in Corollary 5.1. A similar factorization was revealed by Yu. Dyukarev in [24] by developing a multiplicative representation of the RM of the truncated Stieltjes matrix moment (TSMM) problem. Generalized Stieltjes parameters for Nevanlinna-Pick interpolation problems in certain Nevanlinna classes were considered in [28] and [40]. The determinateness of the TSMM problem was obtained in [25] with the help of Dyukarev–Stieltjes matrix parameters of the TSMM. In [26] a criterion for complete indeterminacy of limiting Stieltjes interpolation problem in terms of orthonormal matrix functions was achieved. In [27], by using a decomposition of the RM of the TSMM problem, the following were demonstrated: necessary and sufficient conditions for the TSMM problem to have a unique solution and infinitely many solutions for the Hamburger moment problem with the same moments. Note that in [45] and [15] the operator approach was employed to solve the THMM.

The main results of the present work are the new factorization of the RM  $U^{(m)}$  presented in Theorem 5.2 and the representation of the extremal solutions with the help of matrix continued fractions in terms of DSM parameters; see Theorem 6.1. Matrix continued fractions were studied in [1], [42], [46], [47], [16] and references therein.

Additionally, the following facts are attained:

- (i) We obtain a multiplicative representation of the auxiliary matrices  $\tilde{U}_2^{(m)}$  (as in Corollary 4.1) via new auxiliary Blaschke–Potapov factors  $d^{(m)}$ ; see (4.5), (4.6), (4.7).
- (ii) Each Blaschke–Potapov factor  $d^{(2j)}$  (resp.  $d^{(2j+1)}$ ) is decomposed via new Dyukarev–Stieltjes parameters type matrices  $\mathbf{m}_k$  and  $\mathbf{r}_k$  (resp.  $\mathbf{t}_k$  and  $\mathbf{l}_k$ ). See Theorem 5.1.
- (iii) Explicit relations between  $P_{2,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$ ,  $\Theta_{1,j}$  at  $z = a$  and DSM parameters  $\mathbf{m}_j$ ,  $\mathbf{l}_j$  are given. See Proposition 7.1 and Remark 7.1.

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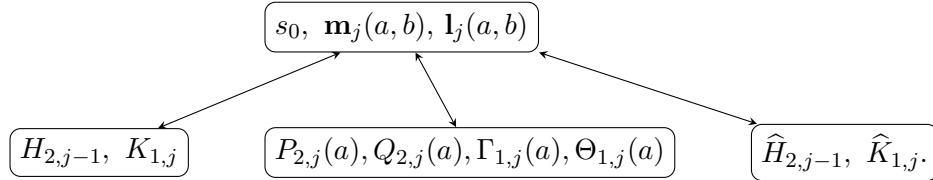
- (iv) Explicit relations between the Schur complements  $\widehat{H}_{2,j}$ ,  $\widehat{K}_{1,j}$  (as in (4.1), (4.2)) and the DSM parameters  $\mathbf{m}_j$ ,  $\mathbf{l}_j$  (as in (2.11), (2.6)) are given. On the other hand, explicit relations between the Schur complement  $\widehat{H}_{2,j}$ ,  $\widehat{K}_{1,j}$  and polynomials  $P_{2,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$ ,  $\Theta_{1,j}$  at  $z = a$  are obtained. See Remark 7.2, Remark 7.3, and Proposition 7.3.
- (v) Given  $s_0$  and a sequence of DSM parameters  $\mathbf{m}_j$ ,  $\mathbf{l}_j$ , we obtain a sequence  $(s_l)_{l=0}^{2j+1}$  (resp.  $(-abs_l + (a+b)s_{l+1} - s_{l+2})_{l=0}^{2j-2}$ ) such that  $K_{1,j}$  (resp.  $H_{2,j-1}$ ) is a positive definite matrix. See Proposition 7.2.

To the author's knowledge, the results contained in (i) through (v) are also new in the scalar case. In Section 8, the scalar version of the DSM parameters  $\mathbf{m}_j$  and  $\mathbf{l}_j$  via determinants is indicated.

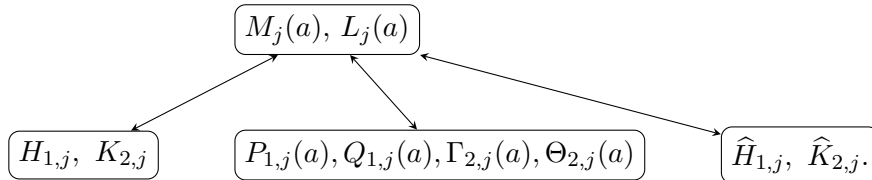
In comparison to the DSM parameters  $M_k$  and  $L_k$  [9], the new DSM parameters  $\mathbf{m}_j$  and  $\mathbf{l}_j$  depend on both terminal points of the interval  $[a, b]$ . Other DSM parameters which also depend on  $a$  and  $b$  were introduced in [5]. In turn the mentioned parameters are different from the ones studied in [9] (also in [4]), where the parameters depend only on  $a$ . In Remark 7.4 by setting  $b \rightarrow +\infty$  and  $a = 0$  in the DSM parameters  $\mathbf{m}_j$  and  $\mathbf{l}_j$ , we obtain the Dyukarev–Stieltjes parameters of the TSMM problem [25].

Throughout the paper we decisively use the forms (1.11) and (1.12) of the RM of the THMM problem obtained in [6] where the elements of the RM are given with the help of four orthogonal polynomials and their second kind polynomials. Orthogonal matrix polynomials (OMP) were first considered by M.G. Krein in 1949 [36], [37]. Further investigations of OMP were made by J.S. Geronimo [32], I.V. Kovalishina [34], [35], H. Dym [23], B. Simon [41], Damanik/Pushnitski/Simon [16] and the references therein. See also [18], [19], [20], [21], [22], [29], [17], [38], [27] and [14].

A brief outline of the relationships between the DSM parameters  $\mathbf{m}_j$ ,  $\mathbf{l}_j$  and the polynomials  $P_{2,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$ ,  $\Theta_{1,j}$  as well as the matrices  $H_{2,j-1}$ ,  $K_{1,j}$ ,  $\widehat{H}_{1,j-1}$ ,  $\widehat{K}_{1,j}$  achieved in the present work is given by



Here  $A \longleftrightarrow B$  means both sides possess an explicit interrelation between  $A$  and  $B$ . The mentioned relationships complete in some sense the ones obtained in [9]:



## 2. Notations and preliminaries

An application of the Hausdorff moment problem method, in particular of the solution set (1.10) with (1.11), (1.12) in the scalar case is used in [8] to solve the admissible bounded control problem of the Brunovsky control system of dimension  $m$  for  $m \geq 2$  via orthogonal polynomials on  $[0, T]$  and their second kind polynomials. See also [12] and [13].

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In this section we include the main notations and objects which we use throughout the paper. The auxiliary RM  $\widehat{U}_2^{(2j)}$  is introduced which is used instead of the auxiliary RM  $\widetilde{U}_2^{(2j)}$ , previously defined in [10, Formula (6.2)].

The orthogonal matrix polynomials  $P_{k,j}$ ,  $\Gamma_{k,j}$  on  $[a, b]$  as well as their second kind polynomials  $Q_{k,j}$ ,  $\Theta_{k,j}$  are recalled. The mentioned matrix polynomials together with the connection between the auxiliary RM  $\widetilde{U}_2^{(2j+1)}$ ,  $\widehat{U}_2^{(2j)}$  and the RM  $U^{(m)}$  play an important role in the present work.

### 2.1. Auxiliary matrices and auxiliary resolvent matrices

Let  $R_j : \mathbb{C} \rightarrow \mathbb{C}^{(j+1)q \times (j+1)q}$  be given by

$$R_j(z) := (I_{(j+1)q} - zT_j)^{-1}, \quad j \geq 0, \quad (2.1)$$

with  $T_0 := 0_q$ ,  $T_j := \begin{pmatrix} 0_{q \times jq} & 0_q \\ I_{jq} & 0_{jq \times q} \end{pmatrix}$ ,  $j \geq 1$ .

Let

$$v_0 := I_q, \quad v_j := \begin{pmatrix} I_q \\ 0_{jq \times q} \end{pmatrix} = \begin{pmatrix} v_{j-1} \\ 0_q \end{pmatrix}, \quad \forall j \geq 0. \quad (2.2)$$

Furthermore, let

$$y_{[j,k]} := \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_k \end{pmatrix}, \quad 0 \leq j \leq k \leq 2n. \quad (2.3)$$

Let

$$\begin{aligned} \widetilde{u}_{1,0} &:= s_0, & \widetilde{u}_{2,0} &:= -s_0, \\ \widetilde{u}_{1,j} &:= y_{[0,j]} - b \begin{pmatrix} 0_q \\ y_{[0,j-1]} \end{pmatrix}, & \widetilde{u}_{2,j} &:= -y_{[0,j]} + a \begin{pmatrix} 0_q \\ y_{[0,j-1]} \end{pmatrix} \end{aligned} \quad (2.4)$$

for every  $1 \leq j \leq n-1$ . In addition, for  $1 \leq j \leq n$  let

$$\widetilde{Y}_{1,j} := by_{[j,2j-1]} - y_{[j+1,2j]}, \quad \widetilde{Y}_{2,j} := -ay_{[j,2j-1]} + y_{[j+1,2j]}. \quad (2.5)$$

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Let  $\widehat{K}_{1,j}$  (resp.  $\widehat{K}_{2,j}$ ) denote the Schur complement of the block  $bs_{2j}-s_{2j+1}$  (resp.  $-as_{2j}+s_{2j+1}$ ) of the matrix  $K_{1,j}$  (resp.  $K_{2,j}$ ). In addition, denote

$$\widehat{K}_{1,0} = bs_0 - s_1, \quad \widehat{K}_{1,j} := bs_{2j} - s_{2j+1} - \widetilde{Y}_{1,j}^* K_{1,j-1}^{-1} \widetilde{Y}_{1,j}, \quad 1 \leq j \leq n, \quad (2.6)$$

$$\widehat{K}_{2,0} = -as_0 + s_1, \quad \widehat{K}_{2,j} := -as_{2j} + s_{2j+1} - \widetilde{Y}_{2,j}^* K_{2,j-1}^{-1} \widetilde{Y}_{2,j}, \quad 1 \leq j \leq n. \quad (2.7)$$

The quantities (2.6) and (2.7) have been defined in [17] for  $a = 0$  and  $b = 1$ .

Let

$$u_{1,0} := 0_q, \quad u_{1,j} := \begin{pmatrix} 0_q \\ -y_{[0,j-1]} \end{pmatrix}, \quad 1 \leq j \leq n$$

and

$$u_{2,0} := -(a+b)s_0 + s_1, \quad u_{2,j} := \begin{pmatrix} u_{2,0} \\ -\widehat{y}_{[0,j-2]} \end{pmatrix}, \quad 1 \leq j \leq 2n. \quad (2.8)$$

Moreover, let

$$\widehat{s}_j := -abs_j + (a+b)s_{j+1} - s_{j+2}, \quad 0 \leq j \leq 2n-2 \quad (2.9)$$

and

$$\widehat{y}_{[j,k]} := \begin{pmatrix} \widehat{s}_j \\ \widehat{s}_{j+1} \\ \vdots \\ \widehat{s}_k \end{pmatrix}, \quad 0 \leq j \leq k \leq 2n-2.$$

Note that by (2.3) and (2.9)

$$\widehat{y}_{[j,k]} = -aby_{[j,k]} + (a+b)y_{[j+1,k+1]} - y_{[j+2,k+2]}.$$

We also denote

$$Y_{1,j} := y_{[j,2j-1]}, \quad 1 \leq j \leq n, \quad Y_{2,j} := \widehat{y}_{[j,2j-1]}, \quad 1 \leq j \leq n-1.$$

Let  $\widehat{H}_{1,j}$  (resp.  $\widehat{H}_{2,j}$ ) denote the Schur complement of the block  $s_{2j}$  (resp.  $\widehat{s}_{2j-2}$ ) of the matrix  $H_{1,j}$  (resp.  $H_{2,j}$ ): denote  $\widehat{H}_{1,0} = s_0$ ,  $\widehat{H}_{2,0} = \widehat{s}_0$  and

$$\widehat{H}_{1,j} := s_{2j} - Y_{1,j}^* H_{1,j-1}^{-1} Y_{1,j}, \quad 1 \leq j \leq n, \quad (2.10)$$

$$\widehat{H}_{2,j} := \widehat{s}_{2j} - Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j}, \quad 1 \leq j \leq n-1. \quad (2.11)$$

The quantities (2.10) and (2.11) have been defined in [17] for  $a = 0$  and  $b = 1$ .

**Definition 2.1.** Let  $[a, b] \subset \mathbb{R}$ . The sequence  $(s_k)_{k=0}^{2j}$  (resp.  $(s_k)_{k=0}^{2j+1}$ ) is called a Hausdorff positive definite sequence if the block Hankel matrices  $H_{1,j}$  and  $H_{2,j-1}$  (resp.  $K_{1,j}$  and  $K_{2,j}$ ) are both positive definite matrices.

In the sequel, we will consider only Hausdorff positive definite sequences. In this case the THMM problem is called a non degenerate THMM problem.



## 2. Notations and preliminaries

**Definition 2.2.** [11, Formula (6.2)] Let  $(s_k)_{k=0}^{2j+1}$  be a Hausdorff positive sequence. The  $2q \times 2q$  matrix polynomial

$$\tilde{U}_2^{(2j+1)}(z, a, b) := \begin{pmatrix} \tilde{\alpha}_2^{(2j+1)}(z) & \tilde{\beta}_2^{(2j+1)}(z) \\ \tilde{\gamma}_2^{(2j+1)}(z) & \tilde{\delta}_2^{(2j+1)}(z) \end{pmatrix}, \quad z \in \mathbb{C}, \quad 1 \leq j \leq n, \quad (2.12)$$

with

$$\begin{aligned} \tilde{\alpha}_2^{(2j+1)}(z, a, b) &:= I_q - (z - a)\tilde{u}_{1,j}^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) v_j, \\ \tilde{\beta}_2^{(2j+1)}(z, a, b) &:= (z - a)\tilde{u}_{1,j}^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) \tilde{u}_{1,j}, \\ \tilde{\gamma}_2^{(2j+1)}(z, a, b) &:= - (z - a) v_j^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) v_j, \end{aligned}$$

and

$$\tilde{\delta}_2^{(2j+1)}(z, a, b) := I_q + (z - a) v_j^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) \tilde{u}_{1,j}$$

is called the second auxiliary matrix of the THMM problem in the case of an even number of moments.

Let

$$B_{2,j} := (b - a)\tilde{u}_{2,j}^* R_j^*(a) K_{2,j}^{-1} R_j(a) \tilde{u}_{2,j}$$

and

$$A_2^{(2j+1)} := \begin{pmatrix} I_q & B_{2,j} \\ 0_q & I_q \end{pmatrix}. \quad (2.13)$$

In [11], the following equality was proved:

$$U^{(2j+1)} = \begin{pmatrix} \frac{1}{b-z} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \tilde{U}_2^{(2j+1)} A_2^{(2j+1)} \begin{pmatrix} (b-z) I_q & 0_q \\ 0_q & I_q \end{pmatrix}.$$

**Definition 2.3.** [10, Formula (6.2)] Let  $(s_k)_{k=0}^{2j}$  be a Hausdorff positive sequence. The  $2q \times 2q$  matrix polynomial

$$\tilde{U}_2^{(2j)}(z, a, b) := \begin{pmatrix} \tilde{\alpha}_2^{(2j)}(z, a, b) & \tilde{\beta}_2^{(2j)}(z, a, b) \\ \tilde{\gamma}_2^{(2j)}(z, a, b) & \tilde{\delta}_2^{(2j)}(z, a, b) \end{pmatrix}, \quad z \in \mathbb{C}, \quad 0 \leq j \leq n-1, \quad (2.14)$$

with

$$\begin{aligned} \tilde{\alpha}_2^{(2j)}(z, a, b) &:= I_q - (z - a) u_{2,j}^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j, \\ \tilde{\beta}_2^{(2j)}(z, a, b) &:= (z - a) u_{2,j}^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) u_{2,j}, \\ \tilde{\gamma}_2^{(2j)}(z, a, b) &:= - (z - a) v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j \end{aligned}$$

and

$$\tilde{\delta}_2^{(2j)}(z, a, b) := I_q + (z - a) v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) u_{2,j},$$

is called the second auxiliary matrix of the THMM problem in the case of an odd number of moments.

## 2. Notations and preliminaries

**Definition 2.4.** Let  $(s_k)_{k=0}^{2j}$  be an odd Hausdorff positive sequence. The  $2q \times 2q$  matrix polynomial

$$\widehat{U}_2^{(2j)}(z, a, b) := \begin{pmatrix} \widehat{\alpha}_2^{(2j)}(z, a, b) & \widehat{\beta}_2^{(2j)}(z, a, b) \\ \widehat{\gamma}_2^{(2j)}(z, a, b) & \widehat{\delta}_2^{(2j)}(z, a, b) \end{pmatrix}, \quad z \in \mathbb{C}, \quad 0 \leq j \leq n-1, \quad (2.15)$$

with

$$\begin{aligned} \widehat{\alpha}_2^{(2j)}(z, a, b) &:= I_q - (z - a)(u_{2,j}^* + z s_0 v_j^*) R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j, \\ \widehat{\beta}_2^{(2j)}(z, a, b) &:= (z - a)(s_0 + (u_{2,j}^* + z s_0 v_j^*) R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) (u_{2,j} + a v_j s_0)), \\ \widehat{\gamma}_2^{(2j)}(z, a, b) &:= -(z - a) v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j \end{aligned}$$

and

$$\widehat{\delta}_2^{(2j)}(z, a, b) := I_q + (z - a) v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) (u_{2,j} + a v_j s_0)$$

is called the second transformed auxiliary matrix of the THMM problem in the case of an odd number of moments. The adjective transformed in the sequel will be omitted.

Let

$$N_{2,j} := -(b - a)^{-1} v_j^* R_j^*(a) H_{1,j}^{-1} R_j(a) v_j \quad (2.16)$$

and

$$A_2^{(2j)} := \begin{pmatrix} I_q & -a s_0 \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ N_{2,j} & I_q \end{pmatrix}. \quad (2.17)$$

**Remark 2.1.** Let  $(s_j)_{j=0}^{2n}$  be a Hausdorff positive sequence, and let  $U^{(2j)}$ ,  $\widetilde{U}_2^{(2j)}$  and  $\widehat{U}_2^{(2j)}$  be as in (1.11), (2.14) and (2.15). The following equalities are valid:

a)

$$\widehat{U}_2^{(2j)}(z) = \begin{pmatrix} I_q & z s_0 \\ 0_q & I_q \end{pmatrix} \widetilde{U}_2^{(2j)}(z) \begin{pmatrix} I_q & -a s_0 \\ 0_q & I_q \end{pmatrix} \quad (2.18)$$

and b)

$$U^{(2j)}(z) = \begin{pmatrix} \frac{1}{(z-a)(b-z)} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \widehat{U}_2^{(2j-2)}(z) \begin{pmatrix} I_q & 0_q \\ N_{2,j} & I_q \end{pmatrix} \begin{pmatrix} (z-a)(b-z) I_q & 0_q \\ 0_q & \frac{b-z}{b-a} I_q \end{pmatrix}. \quad (2.19)$$

*Proof.* Equalities (2.18) and (2.19) readily follow by direct calculations.  $\square$

### 2.2. Orthogonal matrix polynomials on $[a, b]$

Let us reproduce some notions on OMP which were introduced in [14]. Let  $P$  be a complex  $p \times q$  matrix polynomial. For all  $n \in \mathbb{N}_0$ , let

$$Z_n^{[P]} := [A_0, A_1, \dots, A_n],$$

## 2. Notations and preliminaries

where  $(A_j)_{j=0}^\infty$  is the unique sequence of complex  $p \times q$  matrices such that for all  $z \in \mathbb{C}$  the polynomial  $P$  admits the representation  $P(z) = \sum_{j=0}^\infty z^j A_j$ . Furthermore, we denote by  $\deg P := \sup\{j \in \mathbb{N}_0 : A_j \neq 0_{p \times q}\}$  the *degree of  $P$* . Observe that in the case  $P(z) = 0_{p \times q}$  for all  $z \in \mathbb{C}$  we have thus  $\deg P = -\infty$ . If  $k := \deg P \geq 0$ , we refer to  $A_k$  as the *leading coefficient of  $P$* . For all  $k \in \mathbb{N}_0$  and all  $\kappa \in \mathbb{N}_0$  with  $k \leq \kappa$ , let  $\mathbb{Z}_{k,\kappa} := \{n \in \mathbb{N}_0, k \leq n \leq \kappa\}$ .

**Definition 2.5.** Let  $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ , and let  $(s_j)_{j=0}^{2\kappa}$  be a sequence of complex  $q \times q$  matrices. A sequence  $(P_k)_{k=0}^\kappa$  of complex  $q \times q$  matrix polynomials is called a *monic left orthogonal system of matrix polynomials with respect to  $(s_j)_{j=0}^{2\kappa}$*  if the following three conditions are fulfilled:

- (I)  $\deg P_k = k$  for all  $k \in \mathbb{Z}_{0,\kappa}$ ;
- (II)  $P_k$  has the leading coefficient  $I_q$  for all  $k \in \mathbb{Z}_{0,\kappa}$ ;
- (III)  $Z_n^{[P_j]} H_n (Z_n^{[P_k]})^* = 0_{q \times q}$  for all  $j, k \in \mathbb{Z}_{0,\kappa}$  with  $j \neq k$ , where  $n := \max\{j, k\}$ .

**Remark 2.2.** [14, Remark 3.6] Let  $n \in \mathbb{N}_0 \cup \{\infty\}$ , and let  $(s_j)_{j=0}^{2n}$  be a Hausdorff positive definite sequence, i.e., the corresponding Hankel block matrix  $H_n$  is positive definite. Denote by  $(P_k)_{k=0}^n$  the monic left orthogonal system of matrix polynomials with respect to  $(s_j)_{j=0}^{2n}$ . Let  $\sigma$  be a nonnegative Hermitian  $q \times q$  measure on  $\mathbb{R}$  satisfying  $s_j = \int_{[a,b]} t^j d\sigma(t)$  for  $0 \leq j \leq 2n$ . Thus,

$$\int_{[a,b]} P_j d\sigma P_k^* = \begin{cases} \widehat{H}_j, & \text{if } j = k, \\ 0_q, & \text{if } j \neq k \end{cases}$$

for all  $0 \leq j, k \leq n$  where  $\widehat{H}_j$  denotes the Schur complement of  $H_{j-1}$  in  $H_j$ ; see (2.10).

**Definition 2.6.** Let  $(s_k)_{k=0}^{2j}$  be a Hausdorff positive definite sequence. Let

$$\begin{aligned} P_{1,0}(z) &:= I_q, & P_{2,0}(z) &:= I_q, & Q_{1,0}(z) &:= 0_q, & Q_{2,0}(z, a, b) &:= -(u_{2,0} + z s_0), \\ P_{1,j}(z) &:= (-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_j(z) v_j, & & & & & & 1 \leq j \leq n, \end{aligned} \quad (2.20)$$

$$P_{2,j}(z, a, b) := (-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z) v_j, \quad 1 \leq j \leq n-1, \quad (2.21)$$

$$Q_{1,j}(z) := (-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_{1,j}(z) u_{1,j}, \quad 1 \leq j \leq n$$

and

$$Q_{2,j}(z, a, b) := -(-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z) (u_{2,j} + z v_j s_0), \quad 1 \leq j \leq n-1. \quad (2.22)$$

**Definition 2.7.** Let  $K_{k,j}$ ,  $\widetilde{u}_{k,j}$ ,  $\widetilde{Y}_{k,j}$ , for  $k = 1, 2$ ,  $R_j$  and  $v_j$  be as in (1.9), (2.4), (2.5), (2.4), (2.5), (2.1) and (2.2), respectively.

Let  $(s_k)_{k=0}^{2j+1}$  be a Hausdorff positive definite sequence. Let

$$\Gamma_{1,0}(z) := I_q, \quad \Gamma_{2,0}(z) := I_q, \quad \Theta_{1,0}(z) := s_0, \quad \Theta_{2,0}(z) := -s_0$$

### 3. Algebraic identities

for all  $z \in \mathbb{C}$ . For  $k \in \{1, 2\}$  and  $1 \leq j \leq n$ , define

$$\Gamma_{1,j}(z, b) := (-\tilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q) R_j(z) v_j, \quad (2.23)$$

$$\Gamma_{2,j}(z, a) := (-\tilde{Y}_{2,j}^* K_{2,j-1}^{-1}, I_q) R_j(z) v_j, \quad (2.24)$$

$$\Theta_{1,j}(z, b) := (-\tilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q) R_j(z) \tilde{u}_{1,j} \quad (2.25)$$

and

$$\Theta_{2,j}(z, a) := (-\tilde{Y}_{2,j}^* K_{2,j-1}^{-1}, I_q) R_j(z) \tilde{u}_{2,j}, \quad (2.26)$$

for all  $z \in \mathbb{C}$ .

We usually omit the dependence of the polynomials  $P_{k,j}$ ,  $Q_{k,j}$ ,  $\Gamma_{k,j}$  and  $\Theta_{k,j}$  for  $k = 1, 2$  on the parameters  $a$  and  $b$ .

In [3], (resp. [44]) it was proved that polynomials  $P_{k,j}$  (resp.  $\Gamma_{k,j}$ ) for  $k = 1, 2$  are in fact OMP on  $[a, b]$ . In [6] some properties of second kind polynomials  $Q_{k,j}$  and  $\Theta_{k,j}$  for  $k = 1, 2$  were discussed. In [14] explicit interrelations between  $P_{k,j}$ ,  $\Gamma_{k,j}$  and their second kind polynomials were studied.

For the sake of completeness in the following Remark, we reproduce explicit interrelations between the matrices  $\hat{H}_{k,j}$ ,  $\hat{K}_{k,j}$  and the polynomials  $P_{1,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$ ,  $\Theta_{2,j}$  considered in [6, Corollary 3.4] and [6, Corollary 3.10].

**Remark 2.3.** Let  $\hat{H}_{k,j}$ ,  $\hat{K}_{k,j}$ , for  $k = 1, 2$ ,  $P_{1,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$  and  $\Theta_{2,j}$  be as in (2.10), (2.11), (2.6), (2.7) and Definitions 2.6 and 2.7, respectively. The following equalities then hold:

$$\hat{H}_{1,j} = -P_{1,j}(a)\Theta_{2,j}^*(a), \quad \hat{H}_{2,j} = -Q_{2,j}(a)\Gamma_{1,j+1}^*(a), \quad (2.27)$$

$$\hat{K}_{1,j} = \Gamma_{1,j}(a)Q_{2,j}^*(a), \quad \hat{K}_{2,j} = \Theta_{2,j}(a)P_{1,j+1}^*(a). \quad (2.28)$$

### 3. Algebraic identities

In this section we will single out essential identities concerning the block matrices introduced in Section 2:

$$L_{1,n} := (\delta_{j,k+1} I_q)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}} \quad \text{and} \quad L_{2,n} := (\delta_{j,k} I_q)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}},$$

where  $\delta_{j,k}$  is the Kronecker symbol with  $\delta_{j,k} := 1$  if  $j = k$  and  $\delta_{j,k} := 0$  if  $j \neq k$ .

Let

$$\Xi_{1,j}^K := \begin{pmatrix} -K_{1,j-1}^{-1} \tilde{Y}_{1,j} \\ I_q \end{pmatrix}. \quad (3.1)$$

### 3. Algebraic identities

**Remark 3.1.** *The following identities are valid:*

$$v_{j-1} - L_{2,j}^* v_j = 0, \quad (3.2)$$

$$\tilde{u}_{1,j-1} - L_{2,j}^* \tilde{u}_{1,j} = 0, \quad (3.3)$$

$$L_{2,j} - R_j^{*-1}(\bar{z}) L_{2,j} R_{j-1}^{*-1}(\bar{z}) = 0, \quad (3.4)$$

$$L_{2,j} L_{1,j}^* - T_j^* = 0, \quad (3.5)$$

$$H_{1,j} T_j^* - T_j H_{1,j} - \tilde{u}_{1,j} v_j^* + v_j \tilde{u}_{1,j} = 0 \quad (3.6)$$

$$R_{j-1}^{-1}(a) K_{1,j-1} L_{1,j}^* + L_{2,j}^* K_{1,j} T_j^* - L_{2,j}^* T_j K_{1,j} + L_{2,j}^* T_j K_{2,j} R_j^{*-1}(a) = 0, \quad (3.7)$$

$$T_j K_{1,j} \Xi_{1,j}^K = 0. \quad (3.8)$$

*Proof.* Equalities (3.2), (3.3), (3.4), (3.5) are proved by direct calculations. Identity (3.6) was considered in [11, Proposition 2.1]. Identities (3.7) and (3.8) follow by a straightforward calculation.  $\square$

**Remark 3.2.** *The following identities are valid:*

$$u_{1,j}^* + v_j^* H_{1,j} T_j^* = 0, \quad (3.9)$$

$$v_j^* H_{1,j} - v_{j+1}^* H_{1,j+1} L_{2,j+1} = 0, \quad (3.10)$$

$$T_{j+1}^* L_{1,j+1} + (T_{j+1} T_{j+1}^* - I) L_{2,j+1} - L_{2,j+1} L_{1,j} L_{1,j}^* = 0, \quad (3.11)$$

$$T_{j+1}^* L_{1,j+1} T_j^* - T_{j+1}^* L_{2,j+1} L_{1,j} L_{1,j}^* = 0, \quad (3.12)$$

$$T_{j+1}^* L_{2,j+1} - L_{2,j+1} L_{2,j} L_{1,j}^* = 0, \quad (3.13)$$

$$T_{j+1}^* L_{2,j+1} T_j^* - T_{j+1}^* L_{2,j+1} L_{2,j} L_{1,j}^* = 0, \quad (3.14)$$

$$(I - z T_{j+1}^*) L_{2,j+1} (T_j T_j^* - I) - (T_{j+1} T_{j+1}^* - I) L_{2,j+1} = 0, \quad (3.15)$$

$$(I - z T_{j+1}^*) L_{2,j+1} (I + (z - a) T_j^* R_j^*(\bar{z})) - (I - a T_{j+1}^*) L_{2,j+1} = 0, \quad (3.16)$$

$$v_j v_{j+2}^* H_{1,j+2} L_{1,j+2} + L_{2,j+1}^* H_{1,j+1} + L_{1,j+1}^* T_{j+1} H_{1,j+1} = 0, \quad (3.17)$$

$$v_j v_{j+2}^* H_{1,j+2} L_{2,j+2} + L_{2,j+1}^* T_{j+1} \tilde{H}_{2,j+1} - L_{1,j+1}^* \tilde{H}_{1,j+1} = 0, \quad (3.18)$$

$$- T_j L_{1,j+1}^* H_{1,j+1} + L_{2,j+1} T_{1,j+1} \tilde{H}_{1,j+1} + T_j L_{2,j+1}^* T_{j+1} \tilde{H}_{2,j+1} = 0, \quad (3.19)$$

$$- L_{2,j+1} T_{j+1} \tilde{H}_{0,j+1} + T_j L_{2,j+1}^* H_{1,j+1} = 0. \quad (3.20)$$

*Proof.* Identities (3.9)–(3.20) follow by a straightforward calculation.  $\square$

Denote

$$\Xi_{2,j}^H := \begin{pmatrix} -H_{2,j-1}^{-1} Y_{2,j} \\ I_q \end{pmatrix}. \quad (3.21)$$

#### 4. The Blaschke–Potapov factors of the auxiliary matrices

**Proposition 3.1.** *The following identities are valid:*

$$\begin{aligned} & -T_{j+1}^*(L_{1,j+1} - bL_{2,j+1}) - (T_{j+1}T_{j+1}^* - I)L_{2,j+1}R_j^*(a) \\ & + (I - aT_{j+1}^*)L_{2,j+1}(L_{1,j} - bL_{2,j})L_{1,j}^*R_j^*(a) = 0, \end{aligned} \quad (3.22)$$

$$\begin{aligned} & -R_j(a)v_jv_{j+2}^*H_{1,j+2}(L_{1,j+2} - bL_{2,j+2}) + (L_{1,j+1}^* - bL_{2,j+1}^*) \\ & + L_{2,j+1}R_{j+1}(a)T_{j+1}H_{2,j+1} = 0, \end{aligned} \quad (3.23)$$

$$T_jH_{2,j}\Xi_{2,j}^H = 0. \quad (3.24)$$

*Proof.* Identity (3.13) follows from (3.11)–(3.14). Identity (3.22) follows (3.17)–(3.13). We prove equality (3.24). Let  $\lambda_j := (0_q, 0_q, \dots, 0_q, I_q)$  be a  $q \times jq$  matrix. Thus  $T_j = \begin{pmatrix} T_{j-1} & 0_{jq \times q} \\ \lambda_j & 0_q \end{pmatrix}$ . By using the last equality and equality  $H_{2,j} = \begin{pmatrix} H_{2,j-1} & Y_{2,j} \\ Y_{2,j}^* & \hat{s}_{2j} \end{pmatrix}$ , we have

$$\begin{aligned} T_jH_{2,j}\Xi_{2,j}^H &= \begin{pmatrix} T_{j-1} & 0_{jq \times q} \\ \lambda_j & 0_q \end{pmatrix} \begin{pmatrix} H_{2,j-1} & Y_{2,j} \\ Y_{2,j}^* & \hat{s}_{2j} \end{pmatrix} \begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I_q \end{pmatrix} \\ &= \begin{pmatrix} T_{j-1}H_{2,j-1} & T_{j-1}Y_{2,j} \\ \lambda_jH_{2,j-1} & \lambda_jY_{2,j} \end{pmatrix} \begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I_q \end{pmatrix} = \begin{pmatrix} 0_{jq \times q} \\ 0_q \end{pmatrix}. \end{aligned}$$

□

#### 4. The Blaschke–Potapov factors of the auxiliary matrices

In this section we obtain a multiplicative representation (1.15), (1.16) of the second auxiliary matrices  $\tilde{U}_2^{(2n+1)}$  and  $\hat{U}_2^{(2n)}$  via the Blaschke–Potapov factors  $d^{(2j+1)}$  and  $d^{(2j)}$  defined in (4.5)–(4.7).

Since the matrices  $H_{2,j}$  and  $K_{1,j}$  are positive definite matrices for  $0 \leq j \leq n-1$  and  $0 \leq j \leq n$ , respectively, their inverses can be written as

$$H_{2,j}^{-1} = \begin{pmatrix} H_{2,j-1}^{-1} & 0_{jq \times q} \\ 0_{q \times jq} & 0_q \end{pmatrix} + \begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I_q \end{pmatrix} \hat{H}_{2,j}^{-1}(-Y_{2,j}^*H_{2,j-1}^{-1}, I_q) \quad (4.1)$$

and

$$K_{1,j}^{-1} = \begin{pmatrix} K_{1,j-1}^{-1} & 0_{jq \times q} \\ 0_{q \times jq} & 0_q \end{pmatrix} + \begin{pmatrix} -K_{2,j-1}^{-1}\tilde{Y}_{1,j} \\ I_q \end{pmatrix} \hat{K}_{1,j}^{-1}(-\tilde{Y}_{1,j}^*K_{1,j-1}^{-1}, I_q). \quad (4.2)$$

**Proposition 4.1.** *Let the polynomials  $P_{2,j}$  and  $Q_{2,j}$  be as in Definition 2.6 and  $\hat{H}_{2,j}$  be defined as in (2.11). The block elements of the matrix  $\hat{U}_2^{(2j)}(z)$  defined by (2.15) can be written in the form*

$$\begin{aligned} \hat{\alpha}_2^{(2j)}(z) &= \hat{\alpha}_2^{(2j-2)}(z) + (z-a)Q_{2,j}^*(\bar{z})\hat{H}_{2,j}^{-1}P_{2,j}(a), \\ \hat{\beta}_2^{(2j)}(z) &= \hat{\beta}_2^{(2j-2)}(z) + (z-a)Q_{2,j}^*(\bar{z})\hat{H}_{2,j}^{-1}Q_{2,j}(a), \\ \hat{\gamma}_2^{(2j)}(z) &= \hat{\gamma}_2^{(2j-2)}(z) - (z-a)P_{2,j}^*(\bar{z})\hat{H}_{2,j}^{-1}P_{2,j}(a) \end{aligned}$$

#### 4. The Blaschke–Potapov factors of the auxiliary matrices

and

$$\widehat{\delta}_2^{(2j)}(z) = \widehat{\delta}_2^{(2j-2)}(z) - (z - a)P_{2,j}^*(\bar{z})\widehat{H}_{2,j}^{-1}Q_{2,j}(a).$$

*Proof.* Use (2.2), (4.1) and

$$R_j(z) = \left( \frac{R_{j-1}(z)}{(z^j I_q, z^{j-1} I_q, \dots, z I_q)} \middle| \frac{0_{jq \times q}}{I_q} \right), \quad u_{2,j} = \begin{pmatrix} u_{2,j-1} \\ -\widehat{s}_{j-1} \end{pmatrix} \quad (4.3)$$

for  $j \geq 2$ .  $\square$

**Proposition 4.2.** *Let the polynomials  $\Theta_{1,j}$  and  $\Gamma_{1,j}$  be as in Definition 2.7. Then the block elements of the matrix  $\widetilde{U}_2^{(2j+1)}(z)$  defined by (2.12) can be written in the form*

$$\begin{aligned} \widetilde{\alpha}_2^{(2j+1)}(z) &= \widetilde{\alpha}_2^{(2j-1)}(z) - (z - a)\Theta_{1,j}^*(\bar{z})\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a), \\ \widetilde{\beta}_2^{(2j+1)}(z) &= \widetilde{\beta}_2^{(2j-1)}(z) + (z - a)\Theta_{1,j}^*(\bar{z})\widehat{K}_{1,j}^{-1}\Theta_{1,j}(a), \\ \widetilde{\gamma}_2^{(2j+1)}(z) &= \widetilde{\gamma}_2^{(2j-1)}(z) - (z - a)\Gamma_{1,j}^*(\bar{z})\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) \end{aligned}$$

and

$$\widetilde{\delta}_2^{(2j+1)}(z) = \widetilde{\delta}_2^{(2j-1)}(z) + (z - a)\Gamma_{1,j}^*(\bar{z})\widehat{K}_{1,j}^{-1}\Theta_{1,j}(a).$$

*Proof.* Use (2.2), (4.2), the first equality of (4.3) and equality

$$\widetilde{u}_{1,j} = \begin{pmatrix} \widetilde{u}_{1,j-1} \\ -bs_{j-1} + s_j \end{pmatrix} \quad (4.4)$$

for  $j \geq 2$ .  $\square$

**Definition 4.1.** *Let  $\widehat{H}_{2,j}$ ,  $\widehat{K}_{1,j}$ ,  $P_{2,j}$ ,  $Q_{2,j}$ ,  $\Theta_{1,j}$  and  $\Gamma_{1,j}$  be as in (2.11), (2.6), and Definitions 2.6, 2.7, respectively. Define*

$$\begin{aligned} d^{(0)}(z) &:= \begin{pmatrix} I_q & (z - a)s_0 \\ 0_q & I_q \end{pmatrix}, \\ d^{(2j+2)}(z) &:= \begin{pmatrix} I_q + (z - a)Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}P_{2,j}(a) & (z - a)Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a) \\ -(z - a)P_{2,j}^*(a)\widehat{H}_{2,j}^{-1}P_{2,j}(a) & I_q - (z - a)P_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a) \end{pmatrix} \end{aligned} \quad (4.5)$$

(4.6)

for  $0 \leq j \leq n - 1$ , and

$$d^{(2j+1)}(z) := \begin{pmatrix} I_q - (z - a)\Theta_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) & (z - a)\Theta_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Theta_{1,j}(a) \\ -(z - a)\Gamma_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) & I_q + (z - a)\Gamma_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Theta_{1,j}(a) \end{pmatrix} \quad (4.7)$$

for  $0 \leq j \leq n$ .

The matrix function  $d^{(2j)}$  (resp.  $d^{(2j+1)}$ ) is called the Blaschke–Potapov factor of the auxiliary matrix  $\widehat{U}_2^{(2k)}$  (resp.  $\widetilde{U}_2^{(2k+1)}$ ).

#### 4. The Blaschke–Potapov factors of the auxiliary matrices

**Theorem 4.1.** *Let the matrix  $\widehat{U}_2^{(2j)}$  (resp.  $\widetilde{U}_2^{(2j+1)}$ ) be as in (2.15) (resp. (2.12)). Let  $d^{(j)}$  be defined as in (4.5)–(4.7), then*

$$\widehat{U}_2^{(0)}(z) = d^{(0)}(z)d^{(2)}(z), \quad \widetilde{U}_2^{(1)}(z) = d^{(1)}(z), \quad (4.8)$$

$$\widehat{U}_2^{(2j)}(z) = \widehat{U}_2^{(2j-2)}(z)d^{(2j+2)}(z), \quad z \in \mathbb{C}, \quad 1 \leq j \leq n-1 \quad (4.9)$$

and

$$\widetilde{U}_2^{(2j+1)}(z) = \widetilde{U}_2^{(2j-1)}(z)d^{(2j+1)}(z), \quad z \in \mathbb{C}, \quad 1 \leq j \leq n. \quad (4.10)$$

*Proof.* Equality (4.8) readily follows from direct calculation. Now we demonstrate (4.9). Denote

$$G_j^{11}(a) := Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}P_{2,j}(a), \quad (4.11)$$

$$G_j^{12}(a) := Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a), \quad (4.12)$$

$$G_j^{21}(a) := P_{2,j}^*(a)\widehat{H}_{2,j}^{-1}P_{2,j}(a), \quad (4.13)$$

and

$$G_j^{22}(a) := P_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a) \quad (4.14)$$

for  $1 \leq j \leq n-1$ .

Now we prove equality (4.9). By using (4.5), (4.6), (4.11), (4.12), (4.13) and (4.14), Eq. (4.9) can be written in the equivalent form

$$\begin{pmatrix} \widehat{\alpha}_2^{(2j)}(z) & \widehat{\beta}_2^{(2j)}(z) \\ \widehat{\gamma}_2^{(2j)}(z) & \widehat{\delta}_2^{(2j)}(z) \end{pmatrix} - \begin{pmatrix} \widehat{\alpha}_2^{(2j-2)}(z) & \widehat{\beta}_2^{(2j-2)}(z) \\ \widehat{\gamma}_2^{(2j-2)}(z) & \widehat{\delta}_2^{(2j-2)}(z) \end{pmatrix} \begin{pmatrix} I_q + (z-a)G_j^{11}(a) & (z-a)G_j^{12}(a) \\ -(z-a)G_j^{21}(a) & I_q - (z-a)G_j^{22}(a) \end{pmatrix} = 0. \quad (4.15)$$

The left-hand side of (4.15) is equivalent to the following four equalities:

$$\begin{aligned} \Upsilon_{11,j} &:= \widehat{\alpha}_2^{(2j)}(z) - \widehat{\alpha}_2^{(2j-2)}(z) + (z-a) \left( -\widehat{\alpha}_2^{(2j-2)}(z)G_j^{11}(a) + \beta_2^{(2j-2)}(z)G_j^{21}(a) \right), \\ \Upsilon_{12,j} &:= \widehat{\beta}_2^{(2j)}(z) - \widehat{\beta}_2^{(2j-2)}(z) - (z-a) \left( \widehat{\alpha}_2^{(2j-2)}(z)G_j^{12}(a) - \beta_2^{(2j-2)}(z)G_j^{22}(a) \right), \\ \Upsilon_{21,j} &:= \widehat{\gamma}_2^{(2j+1)}(z) - \widehat{\gamma}_2^{(2j-2)}(z) + (z-a) \left( -\widehat{\gamma}_2^{(2j-2)}(z)G_j^{11}(a) + \delta_2^{(2j-2)}(z)G_j^{21}(a) \right) \end{aligned}$$

and

$$\Upsilon_{22,j} := \widehat{\delta}_2^{(2j+1)}(z) - \widehat{\delta}_2^{(2j-2)}(z) - (z-a) \left( \widehat{\gamma}_2^{(2j-2)}(z)G_j^{12}(a) - \delta_2^{(2j-2)}(z)G_j^{22}(a) \right).$$

By taking into account (4.11) and (4.13), we have

$$\begin{aligned} \Upsilon_{11,j} &= (z-a)\widetilde{\Upsilon}_{1,j}\widehat{H}_{2,j}^{-1}P_{2,j}(a), & \Upsilon_{12,j} &= (z-a)\widetilde{\Upsilon}_{1,j}\widehat{H}_{2,j}^{-1}Q_{2,j}(a), \\ \Upsilon_{21,j} &= (z-a)\widetilde{\Upsilon}_{2,j}\widehat{H}_{2,j}^{-1}P_{2,j}(a), & \Upsilon_{22,j} &= (z-a)\widetilde{\Upsilon}_{2,j}\widehat{H}_{2,j}^{-1}Q_{2,j}(a) \end{aligned}$$



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where

$$\begin{aligned}\tilde{\Upsilon}_{1,j+1} &:= Q_{2,j+1}^*(\bar{z}) - Q_{2,j+1}^*(a) + (z-a)(u_{2,j}^* + zs_0v_j^*)R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)v_jQ_{2,j+1}^*(a) \\ &\quad + (z-a)(s_0 + (u_{2,j}^* + zs_0v_j^*)R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)(u_{2,j} + av_js_0))P_{2,j+1}^*(a), \\ \tilde{\Upsilon}_{2,j+1} &:= -P_{2,j+1}^*(\bar{z}) + P_{2,j+1}^*(a) + (z-a)v_j^*R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)v_jQ_{2,j+1}^*(a) \\ &\quad + (z-a)v_j^*R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)(u_{2,j} + av_js_0)P_{2,j+1}^*(a).\end{aligned}$$

Now we verify that

$$\tilde{\Upsilon}_{\ell,j} = 0, \quad \ell, k \in \{1, 2\}, \quad 1 \leq j \leq n-1. \quad (4.16)$$

By using (2.22), (2.21) and (3.21), we have

$$\begin{aligned}\tilde{\Upsilon}_{1,j+1} &= -(u_{2,j+1} + zs_0v_{j+1}^*)R_{j+1}^*(\bar{z}) + (u_{2,j+1} + as_0v_{j+1}^*)R_{j+1}^*(a) + \\ &\quad - (z-a)(u_{2,j} + zs_0v_j^*)R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)(u_{2,j+1}^* + as_0v_{j+1}^*)R_{j+1}^*(a) \\ &\quad + (z-a)\left(s_0 + (u_{2,j}^* + zs_0v_j^*)R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)(u_{2,j} + av_js_0)\right) \\ &\quad \cdot v_{j+1}^*R_{j+1}^*(a) \Xi_{2,j}^H \\ &= (z-a)v_{j+2}^*H_{1,j+2}R_{j+2}^*(\bar{z})\left(-T_{j+2}^*(L_{1,j+2} - bL_{2,j+2})\right. \\ &\quad \left.- ((I - zT_{j+2}^*)L_{2,j+2}(L_{1,j+2} - bL_{2,j+1}) + (z-a)(I - zT_{j+1}^*)L_{2,j+2})\right. \\ &\quad \cdot R_{j+1}^*(\bar{z})T_{j+1}^*(L_{1,j+1} - bL_{2,j+1}))H_{2,j}^{-1}R_j(a)v_jv_{j+2}^*H_{1,j+2} \\ &\quad \cdot (L_{1,j+1} - bL_{2,j+1}) - (I - zT_{j+1}^*)L_{2,j+2}(T_{j+1}T_{j+1}^* - I)R_{j+1}^*(a) \\ &\quad \left. + ((I - zT_{j+2}^*)L_{2,j+2}(L_{1,j+1} - bL_{2,j+1}) + (z-a)(I - zT_{j+2}^*)L_{2,j+2})\right. \\ &\quad \cdot R_{j+1}^*(\bar{z})T_{j+1}^*(L_{1,j+1} - bL_{2,j+1})) \\ &\quad \cdot \left(L_{1,j+1}R_{j+1}^*(a) + H_{2,j}^{-1}(L_{1,j+1}^* - bL_{2,j+1}^*)H_{1,j+1}\right) \Xi_{2,j}^H \\ &= (z-a)v_{j+2}^*H_{1,j+2}R_{j+2}^*(\bar{z})\left(-T_{j+2}^*(L_{1,j+1} - bL_{2,j+1}) - (T_{j+2}T_{j+2}^* - I)\right. \\ &\quad \cdot L_{2,j+2}R_{j+1}^*(a) + (I - aT_{j+1}^*)L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})L_{1,j+1}^*R_{j+1}^*(a) \\ &\quad \left. - (I - aT_{j+2}^*)L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})H_{2,j}^{-1}R_j(a)v_jv_{j+2}^*H_{1,j+2}\right. \\ &\quad \cdot (L_{1,j+1} - bL_{2,j+1}) + (I - aT_{j+2}^*)L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})H_{2,j}^{-1} \\ &\quad \cdot (L_{1,j+1}^* - bL_{2,j+1}^*)H_{1,j+1} \Xi_{2,j}^H \\ &= -(z-a)v_{j+2}^*H_{1,j+2}R_{j+2}^*(\bar{z})(I - aT_{j+2}^*)L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})H_{2,j}^{-1} \\ &\quad \cdot L_{2,j+1}^*T_{j+1}H_{2,j+1}\Xi_{2,j}^H \\ &= 0.\end{aligned}$$

In the second equality we used (3.9) and (3.10). In the third equality we employed (3.15) and (3.16). The penultimate equality follows from (3.22) and (3.23). The last equality follows from identity (3.24). Equality (4.16) for  $\ell = 2$  is proved by using (2.21), (2.22), (3.2), (3.5), (3.23) and (3.24).

## 5. Representation of the RM via DSM parameters of the THMM problem

To prove (4.10), we used the following equalities:

$$\begin{aligned} & \Theta_{1,j}^*(\bar{z}) - \Theta_{1,j}^*(a) + (z-a)\tilde{u}_{1,j-1}^* R_{j-1}^*(\bar{z}) K_{1,j-1}^{-1} R_{j-1}(a) v_{j-1} \Theta_{1,j}^*(a) \\ & - (z-a)\tilde{u}_{1,j-1}^* R_{j-1}^*(\bar{z}) K_{1,j-1}^{-1} R_{j-1}(a) \tilde{u}_{1,j-1} \Gamma_{1,j}^*(a) = 0 \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} & \Gamma_{1,j}^*(\bar{z}) - \Gamma_{1,j}^*(a) + (z-a)\tilde{v}_{j-1}^* R_{j-1}^*(\bar{z}) K_{1,j-1}^{-1} R_{j-1}(a) v_{j-1} \Theta_{1,j}^*(a) \\ & - (z-a)\tilde{v}_{j-1}^* R_{j-1}^*(\bar{z}) K_{1,j-1}^{-1} R_{j-1}(a) \tilde{u}_{1,j-1} \Gamma_{1,j}^*(a) = 0. \end{aligned} \quad (4.18)$$

In turn (4.17), (4.18) are demonstrated by using (2.23), (2.25), (3.3), (3.1), (3.5), (3.2), (3.3), (3.6), (3.7) and (3.8). The theorem is proved.  $\square$

**Corollary 4.1.** *Let the auxiliary matrices  $\tilde{U}_2^{(2j+1)}$  and  $\hat{U}_2^{(2j)}$  defined as in (2.12) and (2.15). Furthermore, let  $d^{(j)}$  be as in (4.5)–(4.7). Thus for  $1 \leq j \leq n$  the admit a Blaschke–Potapov multiplicative representation (1.15) and (1.16).*

The proof follows immediately from Theorem 4.1.

## 5. Representation of the RM via DSM parameters of the THMM problem

In this section we obtain a multiplicative representation of the RM of the THMM problem in terms of OMPs on  $[a, b]$  (more information can be found in Definitions 2.6 and 2.7) and DSM parameters.

**Definition 5.1.** *Let  $a$  and  $b$  be real numbers such that  $a < b$ . Let  $H_{2,j}$ ,  $K_{1,j}$ ,  $R_j$ ,  $v_j$ ,  $u_{2,j}$  be defined by (1.8), (1.9), (2.1), (2.2), (2.4) and (2.8), respectively. Furthermore, let  $H_{2,j}$ ,  $K_{1,j}$  be positive definite matrices. For  $1 \leq j \leq n-1$  denote by*

$$\mathbf{r}_0 := s_0,$$

$$\mathbf{r}_j(a, b) := s_0 + (u_{2,j}^* + as_0 v_j^*) R_j^*(a) H_{2,j}^{-1} R_j(a) (u_{2,j} + av_j s_0), \quad (5.1)$$

$$\mathbf{t}_0(b) := v_0^* R_0^*(a) K_{1,0}^{-1} R_0(a) v_0, \quad \mathbf{t}_j(a, b) := v_j^* R_j^*(a) K_{1,j}^{-1} R_j(a) v_j, \quad (5.2)$$

$$\mathbf{l}_{-1} := s_0, \quad \mathbf{l}_0(a, b) := (u_{2,0}^* + as_0 v_0^*) H_{2,0}^{-1} (u_{2,0} + av_0 s_0), \quad (5.3)$$

$$\begin{aligned} \mathbf{l}_j(a, b) &:= (u_{2,j}^* + as_0 v_j^*) R_j^*(a) H_{2,j}^{-1} R_j(a) (u_{2,j} + av_j s_0) \\ &\quad - (u_{2,j-1}^* + as_0 v_{j-1}^*) R_{j-1}^*(a) H_{2,j-1}^{-1} R_{j-1}(a) (u_{2,j-1} + av_{j-1} s_0), \end{aligned} \quad (5.4)$$

and

$$\mathbf{m}_0(b) := \mathbf{t}_0(b), \quad \mathbf{m}_j(a, b) := v_j^* R_j^*(a) K_{1,j}^{-1} R_j(a) v_j - v_{j-1}^* R_{j-1}^*(a) K_{1,j-1}^{-1} R_{j-1}(a) v_{j-1} \quad (5.5)$$

for  $1 \leq j \leq n$ . The matrices  $\mathbf{l}_j(a, b)$  and  $\mathbf{m}_j(a, b)$  are called the second type Dyukarev–Stieltjes matrix parameters of the THMM problem.

## 5. Representation of the RM via DSM parameters of the THMM problem

Below we shall usually omit the dependence on  $a$  and  $b$  of the matrices (5.1)-(5.5).

Note that from (4.1), (4.2), (4.3), (4.4), (2.22), (2.23), (2.25) and (2.21) the following identities are valid:

$$\mathbf{l}_j = Q_{2,j}^*(a) \widehat{H}_{2,j}^{-1} Q_{2,j}(a), \quad \mathbf{m}_j = \Gamma_{1,j}^*(a) \widehat{K}_{1,j}^{-1} \Gamma_{1,j}(a), \quad (5.6)$$

$$\mathbf{r}_j = \Gamma_{1,j}^{-1}(a) \Theta_{1,j}(a), \quad \mathbf{t}_j = Q_{2,j}^{-1}(a) P_{2,j}(a). \quad (5.7)$$

**Remark 5.1.** Let  $\mathbf{r}_j$ ,  $\mathbf{t}_j$ ,  $\mathbf{l}_j$  and  $\mathbf{m}_j$  be as in (5.1)-(5.5). Thus, the following equalities hold:

$$\mathbf{l}_j = \mathbf{r}_{j+1} - \mathbf{r}_j, \quad j \geq 0, \quad (5.8)$$

$$\mathbf{m}_j = \mathbf{t}_j - \mathbf{t}_{j-1}, \quad j \geq 1. \quad (5.9)$$

Moreover, the matrices  $\mathbf{l}_j$  and  $\mathbf{m}_j$  are positive definite matrices.

*Proof.* Equalities (5.8)-(5.9) follow by direct calculation from (5.1)-(5.5).

By the second equality of (2.27) and the fact that  $\widehat{K}_{1,j}$  and  $\widehat{H}_{2,j}$  are positive definite matrices we obtain that  $\mathbf{l}_j$  and  $\mathbf{m}_j$  also are.  $\square$

The following theorem shows an explicit representation between the Blaschke–Potapov factors  $d^{(j)}$  and the matrices  $\mathbf{r}_j$ ,  $\mathbf{t}_j$ ,  $\mathbf{l}_j$  and  $\mathbf{m}_j$ .

**Theorem 5.1.** Let  $d^{(j)}$  be as in (4.5)-(4.6) and  $\mathbf{r}_j$ ,  $\mathbf{t}_j$ ,  $\mathbf{l}_j$ ,  $\mathbf{m}_j$  be defined by (5.1)-(5.5), respectively. The identity

$$d^{(2j+1)}(z) = \begin{pmatrix} I_q & \mathbf{r}_j \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{m}_j & I_q \end{pmatrix} \begin{pmatrix} I_q & -\mathbf{r}_j \\ 0_q & I_q \end{pmatrix}, \quad (5.10)$$

$$d^{(2j+2)}(z) = \begin{pmatrix} I_q & 0_q \\ -\mathbf{t}_j & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a)\mathbf{l}_j \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ \mathbf{t}_j & I_q \end{pmatrix} \quad (5.11)$$

then holds for  $0 \leq j \leq n$ .

*Proof.* We prove (5.10). For  $j = 0$  the proof can be checked by a direct calculation. Let  $1 \leq j \leq n$ . Denote

$$d^{(2j+1)} := \begin{pmatrix} d_j^{11} & d_j^{12} \\ d_j^{21} & d_j^{22} \end{pmatrix}.$$

The relation (5.10) is equivalent to the following four equalities:

$$d_j^{11} - I_q + (z-a)\mathbf{r}_j\mathbf{m}_j = 0, \quad (5.12)$$

$$d_j^{12} - (z-a)\mathbf{r}_j\mathbf{m}_j\mathbf{r}_j = 0, \quad (5.13)$$

$$d_j^{21} + (z-a)\mathbf{m}_j = 0, \quad (5.14)$$

$$d_j^{22} - I_q - (z-a)\mathbf{m}_j\mathbf{r}_j = 0. \quad (5.15)$$

## 5. Representation of the RM via DSM parameters of the THMM problem

Let us now prove (5.12). By the  $(1, 1)$  element of  $d^{(2j+1)}$ , (5.6) and (5.7), we have

$$\begin{aligned} d_j^{11} - I_q + (z - a)\mathbf{r}_j\mathbf{m}_j \\ = -(z - a)\Theta_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) + (z - a)\Theta_{1,j}^*(a)\Gamma_{1,j}^{*-1}(a)\Gamma_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) \\ = 0 \end{aligned}$$

The equalities (5.13) and (5.15) are proved in a similar way. Note that (5.14) is verified by definition. To prove (5.11) one uses (4.6), the first equality of (5.6) and the second equality of (5.7). Thus theorem 5.1 is proved.  $\square$

Let  $n \in \mathbb{N}_0$ , and let  $A_0, \dots, A_n$  be complex  $q \times q$  matrices. Then let

$$\overrightarrow{\prod_{j=0}^n} A_j = A_0 A_1 \cdots A_{n-1} A_n \quad \text{and} \quad \overleftarrow{\prod_{j=0}^n} A_j = A_n A_{n-1} \cdots A_1 A_0$$

denote the right and left product of the matrices  $A_0, A_1, \dots, A_n$ , respectively.

The following corollary readily yields by employing (5.8), (5.9), Theorem 5.1 and Corollary 4.1.

**Corollary 5.1.** *Let  $\widehat{U}_2^{(2n)}$  and  $\widetilde{U}_2^{(2n+1)}$  be as in (2.15) and (2.12), respectively. Thus, the equalities*

$$\widehat{U}_2^{(2n)} = \overrightarrow{\prod_{k=0}^{n-1}} \left[ \begin{pmatrix} I_q & (z - a)\mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -\mathbf{m}_k & I_q \end{pmatrix} \right] \quad (5.16)$$

and

$$\widetilde{U}_2^{(2n+1)} = \overrightarrow{\prod_{k=0}^n} \left[ \begin{pmatrix} I_q & \mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -(z - a)\mathbf{m}_k & I_q \end{pmatrix} \right] \quad (5.17)$$

are valid.

Now we derive a new representation of the RM of the THMM problem via DSM parameters in both cases for odd and even number of moments. We also reproduce an analogue representation given in [9, Corollary 3]. To this end let us recall the DSM parameters first introduced in [4]:

$$\mathbf{M}_0(a) := s_0^{-1}, \quad \mathbf{L}_0(a) := \widetilde{u}_{2,0}^* K_{2,0}^{-1} \widetilde{u}_{2,0}, \quad (5.18)$$

$$\mathbf{M}_j(a) := v_j^* R_j^*(a) H_j^{-1} R_j(a) v_j - v_{j-1}^* R_{j-1}^*(a) H_{1,j-1}^{-1} R_{j-1}(a) v_{j-1}, \quad (5.19)$$

$$\mathbf{L}_j(a) := \widetilde{u}_{2,j}^* R_j^*(a) K_{2,j}^{-1} R_j(a) \widetilde{u}_{2,j} - \widetilde{u}_{2,j-1}^* R_{j-1}^*(a) K_{2,j-1}^{-1} R_{j-1}(a) \widetilde{u}_{2,j-1}. \quad (5.20)$$

**Theorem 5.2.** *Let  $P_{k,j}$ ,  $Q_{k,j}$ ,  $\Gamma_{k,j}$  and  $\Theta_{k,j}$  be as in Definitions 2.6 and 2.7. Furthermore, let the RM  $U^{(m)}$  of the THMM problem be as in (1.11), (1.12).*

## 6. Extremal solutions of the THMM problem via continued fractions in terms of DSM parameters

a) Let  $\mathbf{l}_k, \mathbf{m}_k$ , be as in Definition (5.3)–(5.5). Thus, the following representation of the resolvent matrix in the case of odd numbers of moments holds

$$U^{(2n)}(z, a, b) = \begin{pmatrix} \frac{1}{(b-z)(z-a)} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \prod_{k=0}^{n-1} \left[ \begin{pmatrix} I_q & (z-a)\mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -\mathbf{m}_k & I_q \end{pmatrix} \right] \\ \cdot \begin{pmatrix} I_q & (z-a)\mathbf{l}_{n-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ Q_{2,n-1}^{-1}(a)P_{2,n-1}(a) + \frac{1}{b-a}\Theta_{2,n}^{-1}(a)\Gamma_{2,n}(a) & I_q \end{pmatrix} \\ \cdot \begin{pmatrix} (b-a)(z-a)I_q & 0_q \\ 0_q & \frac{b-z}{b-a}I_q \end{pmatrix}, \quad (5.21)$$

$$U^{(2n+1)}(z, a, b) = \begin{pmatrix} \frac{1}{b-z} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \prod_{k=0}^n \left[ \begin{pmatrix} I_q & \mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{m}_k & I_q \end{pmatrix} \right] \\ \cdot \begin{pmatrix} I_q & -\Gamma_{1,n}^{-1}(a)\Theta_{1,n}(a) - (b-a)P_{1,n+1}^{-1}(a)Q_{1,n+1}(a) \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} (b-z)I_q & 0_q \\ 0_q & I_q \end{pmatrix}. \quad (5.22)$$

b) Moreover, let  $\mathbf{M}_k, \mathbf{L}_k$  be as in (5.18)–(5.20). Thus the following representations hold:

$$U^{(2n)}(z, a, b) = \prod_{k=0}^{n-1} \left[ \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{M}_k & I_q \end{pmatrix} \begin{pmatrix} I_q & \mathbf{L}_k \\ 0_q & I_q \end{pmatrix} \right] \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{M}_n & I_q \end{pmatrix} \\ \cdot \begin{pmatrix} I_q & Q_{1,n}^*(a)P_{1,n}^{*-1}(a) + \frac{1}{b-a}\Theta_{1,n}^*(a)\Gamma_{1,n}^{*-1}(a) \\ 0_q & I_q \end{pmatrix} \quad (5.23)$$

and

$$U^{(2n+1)}(z, a, b) = \begin{pmatrix} \frac{1}{z-a} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \prod_{k=0}^n \left[ \begin{pmatrix} I_q & 0_q \\ -\mathbf{M}_k & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a)\mathbf{L}_k \\ 0_q & I_q \end{pmatrix} \right] \\ \cdot \begin{pmatrix} I_q & 0_q \\ -\Gamma_{2,n}^*(a)\Theta_{2,n}^{*-1}(a) - (b-a)P_{2,n}^*(a)Q_{2,n}^{*-1}(a) & I_q \end{pmatrix} \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix}. \quad (5.24)$$

*Proof.* We prove part a). Equality (5.21) is proved using (1.13), (2.19), (2.16) and (5.16). In a similar manner one proves equality (5.22) by using (1.14), (5.17) and (2.13). Part b) is proved in [9, Corollary 3].  $\square$

## 6. Extremal solutions of the THMM problem via continued fractions in terms of DSM parameters

As a consequence of the multiplicative representation of  $U^{(2n)}$  and  $U^{(2n+1)}$  as in (5.21) and (5.22), we attain a representation of the extremal solutions of the THMM problem through continued fractions in terms of DSM parameters.

Set  $\frac{A}{B} := AB^{-1}$  for  $A, B \in \mathbb{C}^{q \times q}$  with  $B$  invertible.

6. Extremal solutions of the THMM problem via continued fractions in terms of DSM parameters

**Definition 6.1.** Let be  $P_{k,n}$ ,  $Q_{k,n}$  as in Definition 2.6, and let  $\Theta_{k,n}$ ,  $\Gamma_{k,n}$  be as in Definition 2.7. The following  $q \times q$  matrix valued functions defined for all  $z \in \mathbb{C} \setminus [a, b]$ :

$$s_{\mathcal{K}}^{(2n)}(z) := \frac{\Theta_{2,n}^*(\bar{z})}{(z-a)\Gamma_{2,n}^*(\bar{z})}, \quad s_{\mathcal{F}}^{(2n)}(z) := \frac{\Theta_{1,n}^*(\bar{z})}{(b-z)\Gamma_{1,n}^*(\bar{z})}, \quad (6.1)$$

$$s_{\mathcal{K}}^{(2n+1)}(z) := -\frac{Q_{2,n}^*(\bar{z})}{(z-a)(b-z)P_{2,n}^*(\bar{z})}, \quad s_{\mathcal{F}}^{(2n+1)}(z) := -\frac{Q_{1,n+1}^*(\bar{z})}{P_{1,n+1}^*(\bar{z})} \quad (6.2)$$

are called the even (resp. odd) Krein and Friedrichs extremal solutions of the THMM problem.

Note that a justification of the adjective “extremal” has been not yet given. Such a justification will be presented in a forthcoming work. In [25, Definition 4] the extremal solutions of the Stieltjes matrix moment problem are the terminal matrices of a matrix interval.

To obtain a continued fraction representation of the extremal functions, we write a slightly different multiplicative representation of the RM  $U^{(m)}$ ; see (5.21)–(5.24). Let us introduce some additional notation:

$$\mathbb{I}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & (z-a)\mathbf{l}_k \end{pmatrix}, \quad \mathbf{m}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\mathbf{m}_k \end{pmatrix}, \quad (6.3)$$

$$\mathbb{I}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & \mathbf{l}_k \end{pmatrix}, \quad \mathbf{m}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -(z-a)\mathbf{m}_k \end{pmatrix}, \quad (6.4)$$

$$\mathbb{L}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & \mathbf{L}_k \end{pmatrix}, \quad \mathbf{M}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & -(z-a)\mathbf{M}_k \end{pmatrix}, \quad (6.5)$$

$$\mathbb{L}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & (z-a)\mathbf{L}_k \end{pmatrix}, \quad \mathbf{M}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\mathbf{M}_k \end{pmatrix}. \quad (6.6)$$

and

$$\mathcal{D}_1 := \begin{pmatrix} 0_q & \frac{1}{(b-z)(z-a)}I_q \\ I_q & 0_q \end{pmatrix}, \quad \mathcal{D}_2 := \begin{pmatrix} 0_q & \frac{b-z}{b-a}I_q \\ (b-a)(z-a)I_q & 0_q \end{pmatrix}, \quad (6.7)$$

$$\mathcal{D}_3 := \begin{pmatrix} 0_q & \frac{1}{b-z}I_q \\ I_q & 0_q \end{pmatrix}, \quad \mathcal{D}_4 := \begin{pmatrix} (b-z)I_q & 0_q \\ 0_q & I_q \end{pmatrix},$$

$$\mathcal{D}_5 := \begin{pmatrix} \frac{1}{z-a}I_q & 0_q \\ 0_q & I_q \end{pmatrix}, \quad \mathcal{D}_6 := \begin{pmatrix} 0_q & I_q \\ (z-a)I_q & 0_q \end{pmatrix},$$

$$\mathcal{B}_1^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & Q_{1,n}^*(a)P_{1,n}^{*-1}(a) + \frac{1}{b-a}\Theta_{1,n}^*(a)\Gamma_{1,n}^{*-1}(a) \end{pmatrix},$$

$$\mathcal{B}_1^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\Gamma_{2,n}^*(a)\Theta_{2,n}^{*-1}(a) - (b-a)P_{2,n}^*(a)Q_{2,n}^{*-1}(a) \end{pmatrix},$$

$$\mathcal{B}_2^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & Q_{2,n-1}^{-1}(a)P_{2,n-1}(a) + \frac{1}{b-a}\Theta_{2,n}^{-1}(a)\Gamma_{2,n}(a) \end{pmatrix}, \quad (6.8)$$

$$\mathcal{B}_2^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\Gamma_{1,n}^{-1}(a)\Theta_{1,n}(a) - (b-a)P_{1,n+1}^{-1}(a)Q_{1,n+1}(a) \end{pmatrix}.$$

6. Extremal solutions of the THMM problem via continued fractions in terms of DSM parameters

**Lemma 6.1.** *Let the RM  $U^{(2n)}$  and  $U^{(2n+1)}$  be defined as in (1.11) and (1.12). The identities (1.6), (1.7)*

$$U^{(2n)} = \mathbf{M}_0^{(2n)} \mathbf{L}_0^{(2n)} \dots \mathbf{L}_{n-1}^{(2n)} \mathbf{M}_n^{(2n)} \mathcal{B}_1^{(2n)} \quad (6.9)$$

and

$$U^{(2n+1)} = \mathbf{D}_5 \mathbf{M}_0^{(2n+1)} \mathbf{L}_0^{(2n+1)} \dots \mathbf{M}_n^{(2n+1)} \mathbf{L}_n^{(2n+1)} \mathcal{B}_1^{(2n+1)} \mathfrak{D}_6 \quad (6.10)$$

hold.

*Proof.* We prove (1.6). By using (6.7), (6.3) and (6.8) clearly the following equalities are valid:

$$\begin{aligned} \mathcal{D}_1 \mathbb{I}_{-1}^{(2n)} &= \begin{pmatrix} \frac{1}{(b-z)(z-a)} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a) \mathbf{l}_{-1} \\ 0_q & I_q \end{pmatrix} \\ \mathbf{m}_k^{(2n)} \mathbb{I}_k^{(2n)} &= \begin{pmatrix} I_q & 0_q \\ -\mathbf{m}_k & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a) \mathbf{l}_k \\ 0_q & I_q \end{pmatrix}, \\ \mathcal{B}_2^{(2n)} \mathcal{D}_2 &= \begin{pmatrix} I_q & 0_q \\ Q_{2,n-1}^{-1}(a) P_{2,n-1}(a) + \frac{1}{b-a} \Theta_{2,n}^{-1}(a) \Gamma_{2,n}(a) & I_q \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} (b-a)(z-a) I_q & 0_q \\ 0_q & \frac{b-z}{b-a} I_q \end{pmatrix}. \end{aligned}$$

The latter equalities along with (5.21) imply (1.6). In a similar manner ones proves (1.7), (6.9) and (6.10).  $\square$

Following [16, Page 11], we consider  $\mathbb{C}^{q \times q} \otimes \mathbb{C}^{q \times q}$  as a right module over  $\mathbb{C}^{q \times q}$

For  $\mathfrak{U} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  define the transformation

$$\mathcal{T}_{\mathfrak{U}} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (AX + BY)(CX + DY)^{-1} \\ I_q \end{pmatrix} \quad (6.11)$$

if  $CX + DY$  is invertible.

**Theorem 6.1.** *Let  $Q_{k,j}$ ,  $P_{k,j}$ ,  $\Gamma_{k,j}$  and  $\Theta_{k,j}$  for  $k = 1, 2$  be as in Definition 2.6 and Definition 2.7. The extremal solutions (6.1) and (6.2) of the THMM problem can be*

6. Extremal solutions of the THMM problem via continued fractions in terms of DSM parameters

represented via finite matrix continued fractions:

$$\begin{aligned} & \frac{\Theta_{1,n}^*(\bar{z})}{(b-z)\Gamma_{1,n}^*(\bar{z})} \\ &= \frac{s_0}{b-z} + \frac{I_q}{-(z-a)(b-z)\mathbf{m}_0 + \frac{I_q}{\frac{1}{b-z}\mathbf{l}_0 + \frac{I_q}{\ddots - (z-a)(b-z)\mathbf{m}_{n-1} + \frac{I_q}{\frac{1}{b-z}\mathbf{l}_{n-1}}}}}, \end{aligned} \quad (6.12)$$

$$\begin{aligned} & \frac{\Theta_{2,n}^*(\bar{z})}{(z-a)\Gamma_{1,n}^*(\bar{z})} = - \frac{I_q}{-(z-a)\mathbf{M}_0 + \frac{I_q}{\mathbf{L}_0 + \frac{I_q}{\ddots + L_{n-1} + \frac{I_q}{-(z-a)\mathbf{M}_n}}}}, \end{aligned} \quad (6.13)$$

$$\begin{aligned} - \frac{Q_{1,n+1}^*(\bar{z})}{P_{1,n+1}^*(\bar{z})} &= \frac{I_q}{-(z-a)\mathbf{M}_0 + \frac{I_q}{\mathbf{L}_0 + \frac{I_q}{\ddots + L_{n-1} + \frac{I_q}{-(z-a)\mathbf{M}_n + \mathbf{L}_n^{-1}}}}}, \end{aligned} \quad (6.14)$$

$$\begin{aligned} & \frac{Q_{2,n}^*(\bar{z})}{(z-a)(b-z)P_{2,n}^*(\bar{z})} \\ &= \frac{s_0}{b-z} + \frac{I_q}{-(z-a)(b-z)\mathbf{m}_0 + \frac{I_q}{\frac{1}{b-z}\mathbf{l}_0 + \frac{I_q}{\ddots \cdot \frac{1}{b-z}\mathbf{l}_{n-1} + \frac{I_q}{-(z-a)(b-z)\mathbf{m}_n}}}}. \end{aligned} \quad (6.15)$$

*Proof.* We prove (6.12). We apply the transformation (6.11) at  $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0_q \\ I_q \end{pmatrix}$  to both sides of (5.21), more precisely, on the left side  $\mathfrak{U} = U^{(2n)}$ , however, on the right side  $\mathfrak{U}$  is equal to  $\mathcal{D}_1 \mathfrak{l}_{-1}^{(2n)} \mathfrak{m}_0^{(2n)} \dots \mathfrak{m}_{n-1}^{(2n)} \mathfrak{l}_{n-1}^{(2n)} \mathcal{B}_2^{(2n)} \mathcal{D}_2$ . Taking into account the composition property of the Möbius transformation

$$\mathcal{T}_{U^{(2n)}} = \mathcal{T}_{\mathcal{D}_1} \circ \mathcal{T}_{\mathfrak{m}_0^{(2n)}} \circ \mathcal{T}_{\mathfrak{l}_0^{(2n)}} \circ \dots \circ \mathcal{T}_{\mathfrak{m}_{n-1}^{(2n)}} \mathcal{T}_{\mathcal{B}_2^{(2n)}} \circ \mathcal{T}_{\mathcal{D}_2},$$

we attain the following equality:

$$\begin{aligned} & \left( \begin{array}{c} \frac{1}{b-z}\Theta_{1,n}^*(\bar{z})\Gamma_{1,n}^{-1*}(\bar{z}) \\ I_q \end{array} \right) = \\ & \left( \begin{array}{c} \frac{1}{b-z}\mathbf{l}_{-1} + (-(b-z)(z-a)\mathbf{m}_0 + (\frac{1}{b-z}\mathbf{l}_0 + \dots + (-(b-z)(z-a)\mathbf{m}_n + \frac{1}{b-z}\mathbf{l}_n^{-1})^{-1} \dots)^{-1}) \\ I_q \end{array} \right). \end{aligned}$$



## 7. Relations between the OMP, DSM parameters and Schur complements

Equality (6.12) is proved. Equality (6.13) can be verified applying the Möbius transformation  $\mathcal{T}$  to both sides of (6.9) at  $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} I_q \\ 0_q \end{pmatrix}$ . In a similar way, to prove (6.14) one uses (6.10) at  $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0_q \\ I_q \end{pmatrix}$ . To verify (6.15) we employ the transformation  $\mathcal{T}$  to equality (1.7), more precisely, from one side for  $\mathfrak{U} = U^{(2n+1)} \mathbf{D}_4^{-1}$  on the other side  $\mathfrak{U}$  is equal to  $\mathbf{M}_0^{(2n+1)} \mathbf{L}_0^{(2n+1)} \dots \mathbf{M}_n^{(2n+1)} \mathbf{L}_n^{(2n+1)} \mathcal{B}_1^{(2n+1)} \mathfrak{D}_6$  at  $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} I_q \\ 0_q \end{pmatrix}$ .  $\square$

## 7. Relations between the OMP, DSM parameters and Schur complements

In this section, we obtain explicit relations between the OMP on the interval  $[a, b]$  (as in Definitions 2.6 and 2.7), the DSM parameters (5.3)-(5.5) and the Schur complements  $\widehat{H}_{2,j}$  and  $\widehat{K}_{1,j}$ ; see (2.11) and (2.6).

**Proposition 7.1.** *Let the polynomials  $P_{2,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$  and  $\Theta_{1,j}$  be defined as in Definitions 2.6 and 2.7. Let the DSM parameters  $\mathbf{m}_j$  and  $\mathbf{l}_j$  be as in (5.3)-(5.5). The following identities then hold:*

$$Q_{2,j}(a) = (-1)^j \mathbf{m}_0^{-1} \mathbf{l}_0^{-1} \mathbf{m}_1^{-1} \mathbf{l}_1^{-1} \dots \mathbf{l}_{j-1}^{-1} \mathbf{m}_j^{-1}, \quad 0 \leq j \leq n-1, \quad (7.1)$$

$$\Gamma_{1,j}(a) = (-1)^j \mathbf{m}_0^{-1} \mathbf{l}_0^{-1} \mathbf{m}_1^{-1} \mathbf{l}_1^{-1} \dots \mathbf{l}_{j-1}^{-1}, \quad 1 \leq j \leq n, \quad (7.2)$$

$$P_{2,j}(a) = (-1)^j \mathbf{m}_0^{-1} \mathbf{l}_0^{-1} \mathbf{m}_1^{-1} \mathbf{l}_1^{-1} \dots \mathbf{l}_{j-1}^{-1} \mathbf{m}_j^{-1} (\mathbf{m}_0 + \mathbf{m}_1 + \dots + \mathbf{m}_{j-1} + \mathbf{m}_j), \quad 1 \leq j \leq n-1 \quad (7.3)$$

and

$$\Theta_{1,j}(a) = (-1)^j \mathbf{m}_0^{-1} \mathbf{l}_0^{-1} \mathbf{m}_1^{-1} \mathbf{l}_1^{-1} \dots \mathbf{m}_{j-1}^{-1} \mathbf{l}_{j-1}^{-1} (s_0 + \mathbf{l}_0 + \dots + \mathbf{l}_{j-1}), \quad 1 \leq j \leq n. \quad (7.4)$$

*Proof.* By rewriting the equality (1.13) in the form

$$\begin{pmatrix} (z-a)(b-z)I_q & 0_q \\ 0_q & I_q \end{pmatrix} U^{(2j)} = \begin{pmatrix} \frac{1}{(z-a)(b-z)} I_q & 0_q \\ 0_q & \frac{b-a}{b-z} I_q \end{pmatrix} \widehat{U}_2^{(2j-2)} \begin{pmatrix} I_q & 0_q \\ N_{2,j} & I_q \end{pmatrix},$$

by expanding the right hand side of this equality in powers of  $(z-a)$  and by employing (1.11), (5.16), (2.16), we have

$$\begin{aligned} \frac{b-z}{b-a} \Theta_{2,j}^*(\bar{z}) \Theta_{2,j}^{*-1}(a) &= I_q + \dots + (-1)^j (z-a)^{j+1} s_0 \mathbf{m}_0 \mathbf{l}_0 \dots \mathbf{m}_j \mathbf{l}_j \\ &\quad \cdot \left( P_{2,j-1}^*(a) Q_{2,j-1}^{*-1}(a) + \frac{1}{b-a} \Gamma_{2,j}^*(a) \Theta_{2,j}^{*-1}(a) \right), \end{aligned} \quad (7.5)$$

$$\Theta_{1,j}^*(\bar{z}) \Gamma_{1,j}^{*-1}(a) = s_0 + \mathbf{l}_0 + \dots + \mathbf{l}_{j-1} + \dots + (-1)^j (z-a)^j s_0 \mathbf{m}_0 \mathbf{l}_0 \dots \mathbf{l}_{j-1}, \quad (7.6)$$

## 7. Relations between the OMP, DSM parameters and Schur complements

$$\begin{aligned} \frac{1}{b-a} \Gamma_{2,j}^*(\bar{z}) \Theta_{2,j}^{*-1}(a) = & -(\mathbf{m}_0 + \dots + \mathbf{m}_{j-1}) \\ & + \left( P_{2,j-1}^*(a) Q_{2,j-1}^{*-1}(a) + \frac{1}{b-a} \Gamma_{2,j}^*(a) \Theta_{2,j}^{*-1}(a) \right) \\ & + \dots + (-1)^j (z-a)^j \mathbf{m}_0 \mathbf{l}_0 \dots \mathbf{m}_{j-1} \mathbf{l}_{j-1} \\ & \cdot \left( P_{2,j-1}^*(a) Q_{2,j-1}^{*-1}(a) + \frac{1}{b-a} \Gamma_{2,j}^*(a) \Theta_{2,j}^{*-1}(a) \right), \end{aligned} \quad (7.7)$$

$$\Gamma_{1,j}^*(\bar{z}) \Gamma_{1,j}^{*-1}(a) = I_q + \dots + (-1)^j (z-a)^j \mathbf{m}_0 \mathbf{l}_0 \mathbf{m}_1 \mathbf{l}_1 \dots \mathbf{m}_{j-1} \mathbf{l}_{j-1}. \quad (7.8)$$

Similarly by rewriting (1.14) in the form

$$\begin{pmatrix} (b-z)I_q & 0_q \\ 0_q & I_q \end{pmatrix} U^{(2j+1)} \begin{pmatrix} \frac{1}{b-z}I_q & 0_q \\ 0_q & I_q \end{pmatrix} = \tilde{U}_2^{(2j+1)} A_2^{(2j+1)}$$

and by using (1.12), (5.17), (2.13) and (A.2), we have

$$Q_{2,j}^*(\bar{z}) Q_{2,j}^{*-1}(a) = I_q + \dots + (-1)^{j+1} (z-a)^{j+1} s_0 \mathbf{m}_0 \mathbf{l}_0 \mathbf{m}_1 \mathbf{l}_1 \dots \mathbf{l}_{j-1} \mathbf{m}_j, \quad (7.9)$$

$$\begin{aligned} (z-b) Q_{1,j+1}^*(\bar{z}) P_{1,j+1}^{*-1}(a) = & s_0 + \mathbf{l}_0 + \dots + \mathbf{l}_{j-1} - \Theta_{1,j}^*(a) \Gamma_{1,j}^{*-1}(a) \\ & - (b-a) Q_{1,j+1}^*(a) P_{1,j+1}^{*-1}(a) \\ & + \dots + (-1)^{j+1} (z-a)^j s_0 \mathbf{m}_0 \mathbf{l}_0 \mathbf{m}_1 \mathbf{l}_1 \dots \mathbf{l}_{j-1} \mathbf{m}_j \\ & \cdot (-\Theta_{1,j}^*(a) \Gamma_{1,j}^{*-1}(a) - (b-a) Q_{1,j+1}^*(a) P_{1,j+1}^{*-1}(a)) \\ & + \dots + (-1)^{j+1} (z-j)^{j+1} s_0 \mathbf{m}_0 \mathbf{l}_0 \mathbf{m}_1 \mathbf{l}_1 \dots \mathbf{l}_{j-1} \mathbf{m}_j \\ & \cdot (-\Theta_{1,j}^*(a) \Gamma_{1,j}^{*-1}(a) - (b-a) Q_{1,j+1}^*(a) P_{1,j+1}^{*-1}(a)), \end{aligned} \quad (7.10)$$

$$\begin{aligned} P_{2,j}^*(\bar{z}) Q_{2,j}^{*-1}(a) = & \mathbf{m}_0 + \mathbf{m}_1 + \dots + \mathbf{m}_{j-1} \\ & + \dots + (-1)^j (z-a)^j \mathbf{m}_0 \mathbf{l}_0 \mathbf{m}_1 \mathbf{l}_1 \dots \mathbf{l}_{j-1} \mathbf{m}_j, \end{aligned} \quad (7.11)$$

$$\begin{aligned} P_{1,j+1}^*(\bar{z}) P_{1,j+1}^{*-1}(a) = & I_q + \dots + (-1)^{j+1} (z-a)^{j+1} \mathbf{m}_0 \mathbf{l}_0 \mathbf{m}_1 \mathbf{l}_1 \dots \mathbf{l}_{j-1} \mathbf{m}_j \\ & \cdot (-\Theta_{1,j}^*(a) \Gamma_{1,j}^{*-1}(a) - (b-a) Q_{1,j+1}^*(a) P_{1,j+1}^{*-1}(a)). \end{aligned} \quad (7.12)$$

Equalities (7.1), (7.2), (7.3) and (7.4) follow from (7.6), (7.8) and (7.10). Employ the fact that  $P_{2,j}$ ,  $\Gamma_{1,j}$  are monic polynomials.  $\square$

Note that from (7.12) and (7.7), one obtains the following identities:

$$(b-a) Q_{1,j+1}^*(a) - Q_{2,j}(a) + P_{1,j+1}(a) \Gamma_{1,j}^{-1}(a) \Theta_{1,j}(a) = 0, \quad (7.13)$$

$$\Gamma_{1,j}(a) - \Gamma_{2,j}(a) - (b-a) \Theta_{2,j}(a) Q_{2,j-1}^{-1}(a) P_{2,j-1}(a) = 0. \quad (7.14)$$

The following remark is verified by using identities from Proposition 7.1 and (7.13), (7.14).

## 7. Relations between the OMP, DSM parameters and Schur complements

**Remark 7.1.** *The following identities hold:*

$$\begin{aligned} Q_{2,j}(a) &= \Gamma_{1,j}(a) \mathbf{m}_j^{-1}, \\ \Gamma_{1,j}(a) &= -Q_{2,j-1}(a) \mathbf{l}_{j-1}^{-1}, \\ P_{2,j}(a) &= Q_{2,j}(a) (\mathbf{m}_0 + \mathbf{m}_1 + \dots + \mathbf{m}_{j-1}), \\ \Theta_{1,j}(a) &= \Gamma_{1,j}(a) (s_0 + \mathbf{l}_0 + \dots + \mathbf{l}_{j-1}), \end{aligned}$$

and

$$\begin{aligned} Q_{2,j}(a) &= -Q_{2,j-1}(a) \mathbf{l}_{j-1}^{-1} \mathbf{m}_j^{-1}, \\ \Theta_{1,j}(a) &= Q_{2,j-1} \mathbf{l}_{j-1}^{-1} (s_0 + \mathbf{l}_0 + \dots + \mathbf{l}_{j-1}). \end{aligned}$$

**Remark 7.2.** *Let  $P_{1,j}$ ,  $Q_{2,j}$ ,  $\Gamma_{1,j}$  and  $\Theta_{2,j}$  be as in Definitions 2.6 and 2.7. Let  $\widehat{H}_{1,j}$ ,  $\widehat{K}_{2,j}$  be defined by (2.10), (2.11) and  $\mathbf{m}_j$ ,  $\mathbf{l}_j$  be as in (5.3)-(5.5). Therefore, the following identities hold:*

$$\widehat{H}_{2,0} = (\mathbf{m}_0 \mathbf{l}_0)^{*^{-1}} \mathbf{l}_0 (\mathbf{m}_0 \mathbf{l}_0)^{-1}, \quad \widehat{K}_{1,0} = \mathbf{m}_0^{-1} \quad (7.15)$$

and

$$\widehat{H}_{2,j} = \prod_{k=0}^{\overrightarrow{j}} (\mathbf{m}_k \mathbf{l}_k)^{-1*} \mathbf{l}_j \prod_{k=0}^{\overleftarrow{j-1}} (\mathbf{m}_k \mathbf{l}_k)^{-1}, \quad (7.16)$$

$$\widehat{K}_{1,j} = \prod_{k=0}^{\overrightarrow{j}} (\mathbf{m}_k \mathbf{l}_k)^{-1*} \mathbf{m}_j^{-1} \prod_{k=0}^{\overleftarrow{j}} (\mathbf{m}_k \mathbf{l}_k)^{-1}. \quad (7.17)$$

Moreover,

$$\begin{aligned} P_{1,j}(a) &= (-1)^j \widehat{K}_{2,j-1} \widehat{H}_{1,j-1}^{-1} \dots \widehat{K}_{2,0} \widehat{H}_{1,0}^{-1}, \\ \Gamma_{1,j}(a) &= (-1)^j \widehat{H}_{2,j-1} \widehat{K}_{1,j-1}^{-1} \dots \widehat{H}_{1,1} \widehat{H}_{2,0} \widehat{K}_{1,0}^{-1}, \\ Q_{2,j}(a) &= (-1)^j \widehat{K}_{1,j} \widehat{H}_{2,j-1}^{-1} \widehat{K}_{1,j-1} \dots \widehat{H}_{2,0}^{-1} \widehat{K}_{1,0}, \\ \Theta_{2,j}(a) &= (-1)^{j+1} \widehat{H}_{1,j} \widehat{K}_{2,j-1}^{-1} \widehat{H}_{1,j-1} \dots \widehat{K}_{2,0}^{-1} \widehat{H}_{1,0}. \end{aligned}$$

Under the same conditions as the previous remark, the following result is immediately implied.

**Remark 7.3.** *The following identities hold:*

$$\mathbf{m}_0 = \widehat{K}_{1,0}^{-1}, \quad \mathbf{l}_0 = (\widehat{H}_{2,0}^{-1} \widehat{K}_{1,0})^* \widehat{H}_{2,0} (\widehat{H}_{2,0}^{-1} \widehat{K}_{1,0}^{-1})$$

and

$$\begin{aligned} \mathbf{m}_j &= \prod_{k=0}^{\overrightarrow{j-1}} (\widehat{H}_{2,k} \widehat{K}_{1,k}^{-1})^* \widehat{K}_{1,j}^{-1} \prod_{k=0}^{\overleftarrow{j-1}} (\widehat{H}_{2,k} \widehat{K}_{1,k}^{-1}), \\ \mathbf{l}_j &= \prod_{k=0}^{\overrightarrow{j}} (\widehat{H}_{2,k}^{-1} \widehat{K}_{1,k})^* \widehat{H}_{2,j} \prod_{k=0}^{\overleftarrow{j}} (\widehat{H}_{2,k}^{-1} \widehat{K}_{1,k}). \end{aligned}$$

## 7. Relations between the OMP, DSM parameters and Schur complements

**Proposition 7.2.** *Let  $a$  and  $b$  be real numbers such that  $a < b$ . Furthermore, let  $s_0, \mathbf{m}_0(b), (\mathbf{m}_j(a, b))_{j=0}^{n-1}$  and  $(\mathbf{l}_j(a, b))_{j=0}^n$  (resp.  $s_0, (\mathbf{m}_j(a, b))_{j=0}^n$  and  $(\mathbf{l}_j(a, b))_{j=0}^n$ ) be sequences of positive Hermitian complex  $q \times q$  matrices. Let  $(s_j)_{j=0}^{2n}$  (resp.  $(s_j)_{j=0}^{2n+1}$ ) be a sequence recursively defined by*

$$bs_{2j} - s_{2j+1} := \begin{cases} \mathbf{m}_0^{-1}, & \text{if } j = 0, \\ \tilde{Y}_{1,j}^* K_{1,j-1}^{-1} \tilde{Y}_{1,j} + \overrightarrow{\prod_{k=0}^j (\mathbf{m}_k \mathbf{l}_k)^{-1}} \mathbf{m}_j^{-1} \overleftarrow{\prod_{k=0}^j (\mathbf{m}_k \mathbf{l}_k)^{-1}} & \text{if } j \geq 1 \end{cases}$$

and

$$\hat{s}_{2j} := \begin{cases} (\mathbf{m}_0 \mathbf{l}_0)^{-1} \mathbf{l}_0 (\mathbf{m}_0 \mathbf{l}_0), & \text{if } j = 0, \\ Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j} + \overrightarrow{\prod_{k=0}^j (\mathbf{m}_k \mathbf{l}_k)^{-1}} \mathbf{l}_j \overleftarrow{\prod_{k=0}^{j-1} (\mathbf{m}_k \mathbf{l}_k)^{-1}} & \text{if } j \geq 1 \end{cases}$$

for  $j = 0, \dots, n$  (resp.  $j = 0, \dots, n-1$ ). Thus,  $K_{1,j} = \{bs_{k+j} - s_{k+j+1}\}_{k,j=0}^n$  (resp.  $H_{2,j} = \{\hat{s}_{k+j}\}_{k,j=0}^{n-1}$ ) is a positive Hermitian matrix.

*Proof.* By (7.15), (7.16), (7.17) and the fact that  $\mathbf{m}_j$  and  $\mathbf{l}_j$  are positive definite, then

$$\hat{K}_{1,j} > 0, \quad j \in \mathbb{N}_0, \quad (7.18)$$

$$\hat{H}_{2,j} > 0, \quad j \in \mathbb{N}_0. \quad (7.19)$$

Let

$$K_{1,n} = \begin{pmatrix} K_{1,n-1} & \tilde{Y}_{1,n} \\ \tilde{Y}_{1,n}^* & bs_{2n} - s_{2n+1} \end{pmatrix}, \quad H_{2,n-1} = \begin{pmatrix} H_{2,n-2} & Y_{2,n-1} \\ Y_{2,n-1}^* & \hat{s}_{2n-2} \end{pmatrix}. \quad (7.20)$$

In view of (7.18) and the first equality of (7.20) (resp. (7.19) and the second equality of (7.20)), as well as Lemma A.1, we obtain that  $K_{1,j}$  (resp.  $H_{2,j}$ ) is positive definite.  $\square$

**Proposition 7.3.** *a) Let the polynomials  $(Q_{2,j})_{j=0}^{n-1}$  and  $(\Gamma_{1,j})_{j=1}^n$  be as in (2.22) and (2.23), respectively. Thus, the following equalities hold:*

$$\mathbf{m}_0 = Q_{2,0}^{-1}(a), \quad \mathbf{m}_j = Q_{2,j}^{-1}(a) \Gamma_{1,j}(a), \quad 0 \leq j \leq n-1, \quad (7.21)$$

$$\mathbf{l}_j = -\Gamma_{1,j+1}(a)^{-1} Q_{2,j}(a), \quad 0 \leq j \leq n-1. \quad (7.22)$$

*b) Let the polynomials  $(P_{1,j})_{j=1}^{n+1}$ , and  $(\Theta_{2,j})_{j=0}^n$  be as in (2.20) and (2.26), respectively. Thus, the following equalities hold:*

$$\mathbf{M}_0(a) = -\Theta_{2,0}^{-1}(a), \quad \mathbf{M}_j(a) = -\Theta_{2,j}^{-1}(a) P_{1,j}(a), \quad 1 \leq j \leq n, \quad (7.23)$$

$$\mathbf{L}_j(a) = P_{1,j+1}^{-1}(a) \Theta_{2,j}(a), \quad 0 \leq j \leq n. \quad (7.24)$$

*Proof.* Equalities (7.21) and (7.22) readily follow from (7.1) and (7.2), respectively. Equalities (7.23) and (7.24) readily follow from

$$P_{1,j}(a) = (-1)^j \mathbf{M}_0^{-1} \mathbf{L}_0^{-1} \mathbf{M}_1^{-1} L_1^{-1} \dots \mathbf{M}_{j-1}^{-1} \mathbf{L}_{j-1}^{-1}, \quad 1 \leq j \leq n+1,$$

and

$$\Theta_{2,j}(a) = (-1)^{j+1} \mathbf{M}_0^{-1} \mathbf{L}_0^{-1} \mathbf{M}_1^{-1} L_1^{-1} \dots \mathbf{L}_{j-1}^{-1} \mathbf{M}_j^{-1}, \quad 1 \leq j \leq n$$

which were proved in [9, Proposition 4.1].  $\square$

## 8. Scalar case

In [25] Dyukarev created the Stieltjes parameters for the Stieltjes matrix moment problem which in our notations are given by  $\mathbf{M}_0(0)$ ,  $\mathbf{L}_0(0)$ ,  $\mathbf{M}_j(0)$  and  $\mathbf{L}_j(0)$ . See also [7].

The following remark gives the interrelation between the mentioned Stieltjes parameters and the DSM parameters studied in the present work.

**Remark 7.4.** *Let  $\mathbf{M}_j$  and  $\mathbf{L}_j$  be the DSM parameters as in (5.18)-(5.19). Furthermore, let the DSM parameters be as in (5.3), (5.4) and (5.5). Thus, the following relations are valid:*

$$\mathbf{M}_j(0) = \lim_{b \rightarrow +\infty} b \mathbf{m}_j(0, b), \quad \mathbf{L}_j(0) = \lim_{b \rightarrow +\infty} b^{-1} \mathbf{l}_j(0, b). \quad (7.25)$$

This Remark can be verified by direct calculations.

## 8. Scalar case

This section is related to the scalar version of the DSM parameters  $\mathbf{m}_j$  and  $\mathbf{l}_j$ .

For  $q = 1$ , let  $n \in \mathbb{N}$ , and let  $(s_j)_{j=0}^{2n}$  (resp.  $(s_j)_{j=0}^{2n+1}$ ) be a Hausdorff positive definite sequence of numbers. Let

$$D_{x,j}^{(3)} := \begin{pmatrix} s_0^{(3)} & s_1^{(3)} & \dots & s_j^{(3)} \\ s_1^{(3)} & s_2^{(3)} & \dots & s_{j+1}^{(3)} \\ \dots & \dots & \dots & \dots \\ s_{j-1}^{(3)} & s_j^{(3)} & \dots & s_{2j-1}^{(3)} \\ 1 & x & \dots & x^j \end{pmatrix}, \quad (8.1)$$

with  $s_j^{(3)} := bs_j - s_{j+1}$ . Furthermore, let

$$E_{x,j}^{(2)} := \begin{pmatrix} s_0^{(2)} & s_1^{(2)} & \dots & s_j^{(2)} \\ s_1^{(2)} & s_2^{(2)} & \dots & s_{j+1}^{(2)} \\ \dots & \dots & \dots & \dots \\ s_{j-1}^{(2)} & s_j^{(2)} & \dots & s_{2j-1}^{(2)} \\ e_{2,0}(x) & e_{2,1}(x) & \dots & e_{2,j}(x) \end{pmatrix}, \quad (8.2)$$

where

$$(e_{2,0}(x), e_{2,1}(x), \dots, e_{2,j}(x)) := -(u_{2,j}^* + xv_j^* s_0) R_j^*(x).$$

The matrices  $D_{x,j}^{(2)}$  and  $E_{x,j}^{(2)}$  were introduced in [6, Subsection 2.4]. The following remark is related to the scalar version of the DSM parameters  $\tilde{m}_j$  and  $\tilde{l}_j$ .

**Remark 8.1.** *Let  $\tilde{m}_j$  and  $\tilde{l}_j$  be the scalar version of the DSM parameters  $\mathbf{l}_j$  and  $\mathbf{m}_j$  defined by (5.3)-(5.5). Let  $D_{x,j}^{(3)}$  and  $E_{x,j}^{(2)}$  be as in (8.1) and (8.2), respectively. Thus,*

## A. Appendix

the following identities hold:

$$\tilde{l}_0(a, b) = \frac{(E_{a,0}^{(2)})^2}{H_{2,0}}, \quad \tilde{m}_j(b) = \frac{1}{bs_0 - s_1}, \quad (8.3)$$

$$\tilde{l}_j(a, b) = \frac{(\det E_{a,j}^{(2)})^2}{\det H_{2,j} \det H_{2,j-1}}, \quad \tilde{m}_j(a, b) = \frac{(\det D_{a,j}^{(3)})^2}{\det K_{1,j} \det K_{1,j-1}}, \quad j \geq 1. \quad (8.4)$$

*Proof.* The first and second equality of (8.3) are an immediate consequence of (5.6) for  $q = 1$ . The first equality of (8.4) is proved by employing (5.6) for  $q = 1$ , [6, Equation (2.40)] and the relation

$$\widehat{H}_{2,j} = \frac{\det H_{2,j}}{\det H_{2,j-1}}, \quad j \geq 1.$$

The second equality of (8.4) can be proved in a similar way. Use (5.6) for  $q = 1$ , [6, Equation (2.38)] and equality  $\widehat{K}_{1,j} = \frac{\det K_{1,j}}{\det K_{1,j-1}}$ .  $\square$

Note that scalar Stieltjes parameters of Remark 8.1 coincide with (1.2) as  $b$  approaches infinity and  $a = 0$ .

## A. Appendix

In this appendix we reproduce some results from [9], which are used in the present work.

**Definition A.1.** [10, Formula (6.2)] Let  $(s_k)_{k=0}^{2j}$  be an odd Hausdorff positive on  $[a, b]$  sequence. The  $2q \times 2q$  matrix polynomial

$$\tilde{U}_2^{(2j)}(z, a, b) := \begin{pmatrix} \tilde{\alpha}_2^{(2j)}(z, a, b) & \tilde{\beta}_2^{(2j)}(z, a, b) \\ \tilde{\gamma}_2^{(2j)}(z, a, b) & \tilde{\delta}_2^{(2j)}(z, a, b) \end{pmatrix}, \quad z \in \mathbb{C}, \quad 0 \leq j \leq n-1,$$

with

$$\begin{aligned} \tilde{\alpha}_2^{(2j)}(z, a, b) &:= I_q - (z - a)u_{2,j}^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j, \\ \tilde{\beta}_2^{(2j)}(z, a, b) &:= (z - a)u_{2,j}^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) u_{2,j}, \\ \tilde{\gamma}_2^{(2j)}(z, a, b) &:= -(z - a)v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j \end{aligned}$$

and

$$\tilde{\delta}_2^{(2j)}(z, a, b) := I_q + (z - a)v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) u_{2,j}$$

is called the second auxiliary matrix of the THMM problem in the case of an odd number of moments.

## References

**Remark A.1.** [9, Equalities (1.30) and (1.31)] The following identities are valid:

$$\Gamma_{2,j}^*(\bar{z}, a) \Theta_{2,j}^{*-1}(a, a) = -v_j^* R_j^*(\bar{z}) H_{1,j}^{-1} R_j(a) v_j, \quad (\text{A.1})$$

$$Q_{1,j+1}^*(\bar{z}) P_{1,j+1}^{*-1}(a) = -\tilde{u}_{2,j}^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) \tilde{u}_{2,j}. \quad (\text{A.2})$$

Finally, let us recall the following well-known result below.

**Lemma A.1.** [2, Proposition 8.2.4] Let  $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$  be a Hermitian  $(n+m) \times (n+m)$  matrix. Therefore, the following statements are equivalent:

- i)  $A > 0$ .
- ii)  $A_{11} > 0$  and  $A_{12}^* A_{11}^{-1} A_{12} < A_{22}$ .
- iii)  $A_{22} > 0$  and  $A_{12} A_{22}^{-1} A_{12}^* < A_{11}$ .

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